Cohomological Methods in Algebraic Geometry

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Contents

CHAPTER 1

Some (less essential) basic concepts of category theory

1.1 Categories

Definition 1 Category.

A **category** $\mathcal{C}$ consists of the following data:

1. A class of **objects**, denoted by $\mathrm{Ob}(\mathcal{C})$.
2. For any pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set of **morphisms** (or arrows), denoted by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.
3. For any three objects X, Y, Z, a **composition law**:

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z), \quad (g, f) \longmapsto g \circ f.$$

These data must satisfy the following axioms:

i. **Associativity:** For $f \in \mathrm{Hom}(X, Y)$, $g \in \mathrm{Hom}(Y, Z)$, and $h \in \mathrm{Hom}(Z, W)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

ii. **Identity:** For every object X, there exists a unique morphism $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ such that for all $f : W \to X$ and $g : X \to Y$,

$$\mathrm{id}_X \circ f = f \quad \text{and} \quad g \circ \mathrm{id}_X = g.$$

Example 2 Key Categories in this Book.

Presented here are some of the fundamental categories frequently used in this book. They are not exhaustive, but they constitute the most basic elements.

i. Set: The category of sets and functions.

ii. Ab: The category of abelian groups and group homomorphisms.

iii. Ring: The category of commutative rings with unity and ring homomorphisms.

iv. Mod_A: The category of modules over a ring A.

v. Top: The category of topological spaces and continuous maps.

vi. Sch: The category of schemes and morphisms of schemes (to be defined).

Definition 3 Opposite Category.

For any category $\mathcal{C}$, the **opposite category** $\mathcal{C}^{\text{op}}$ has the same objects as $\mathcal{C}$, but arrows are reversed:

$$\mathrm{Hom}_{\mathcal{C}^{\text{op}}}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(Y,X).$$

This concept is crucial for algebraic geometry, as geometry often behaves "oppositely" to algebra (e.g., larger ideals define smaller closed sets).

Functors are the structure-preserving maps between categories.

Definition 4 Functor.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. A **covariant functor** $F:\mathcal{C}\to\mathcal{D}$ assigns to each object $X\in\mathcal{C}$ an object $F(X)\in\mathcal{D}$, and to each morphism $f:X\to Y$ a morphism $F(f):F(X)\to F(Y)$ in $\mathcal{D}$, preserving identity and composition:

$$F(\mathrm{id}_X)=\mathrm{id}_{F(X)},\quad F(g\circ f)=F(g)\circ F(f).$$

2. A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor $F:\mathcal{C}^{\text{op}}\to\mathcal{D}$. Explicitly, it reverses arrows: if $f:X\to Y$, then $F(f):F(Y)\to F(X)$, satisfying $F(g\circ f)=F(f)\circ F(g)$.

Example 5 The Bridge between Algebra and Geometry.

These are two classical covariant functors from algebraic geometry. We will define them in the following section; beginners can safely omit this material.

i. The **Spectrum functor** is a contravariant functor:

$$\mathrm{Spec}:\mathsf{Ring}\longrightarrow\mathsf{Top}\quad(\text{or } \mathsf{Sch}).$$

A ring homomorphism $A\to B$ induces a continuous map $\mathrm{Spec}B\to\mathrm{Spec}A$.

ii. The **Global Sections functor** is a covariant functor:

$$\Gamma:\mathsf{Sch}\longrightarrow\mathsf{Ring},\quad(X,\mathscr{O}_X)\longmapsto\Gamma(X,\mathscr{O}_X).$$

Example 6 Presheaves as Functors.

Similarly, the definition of a presheaf is itself a classical contravariant functor. We will revisit this concept in the next section.

Let X be a topological space. The open sets of X form a category $\mathsf{Op}(X)$ where morphisms are inclusions. A **presheaf** $\mathscr{F}$ on X is simply a contravariant functor:

$$\mathscr{F} : \mathsf{Op}(X)^{\mathrm{op}} \longrightarrow \mathsf{Set} \quad (\text{or } \mathsf{Ab}, \mathsf{Ring}).$$

If categories are mathematical structures and functors are maps between them, then natural transformations are maps between functors.

Definition 7 Natural Transformation.

Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A **natural transformation** $\eta : F \to G$ consists of a family of morphisms in $\mathcal{D}$,

$$\{\eta_X : F(X) \to G(X)\}_{X \in \mathrm{Ob}(\mathcal{C})},$$

such that for every morphism $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\eta_Y"] & G(Y)
\end{tikzcd}$$

If every component η_X is an isomorphism in $\mathcal{D}$, then η is called a **natural isomorphism**, denoted $F \cong G$.

Note 8 Why is this important?.

The importance of these categories in algebraic geometry lies in the fact that they provide us with a structured framework for research.

i. **Morphisms of Presheaves:** As we shall see in Chapter 1, a morphism between presheaves $\phi : \mathscr{F} \to \mathscr{G}$ is precisely a natural transformation between the functors $\mathscr{F}$ and $\mathscr{G}$. The commutativity of the diagram ensures compatibility with restriction maps.

ii. **Equivalence of Categories:** We say two categories $\mathcal{C}$ and $\mathcal{D}$ are **equivalent** if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \cong \mathrm{id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ (natural isomorphisms). This is the rigorous language behind our theorem $\mathsf{Mod}_A \simeq \mathsf{QCoh}(\mathrm{Spec} A)$.

In category theory, we define properties of morphisms not by looking at elements (which may not exist), but by their interaction with other morphisms (cancellation properties).

Definition 9 Monomorphism and Epimorphism.

Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$.

1. f is a **monomorphism** (or **mono**) if it is left-cancellable. For any object Z and any morphisms $g_1, g_2 : Z \to X$:

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

(Notation: $X \hookrightarrow Y$).

2. f is an **epimorphism** (or **epi**) if it is right-cancellable. For any object Z and any morphisms $h_1, h_2 : Y \to Z$:

$$h_1 \circ f = h_2 \circ f \implies h_1 = h_2.$$

(Notation: $X \twoheadrightarrow Y$).

Definition 10 Isomorphism.

A morphism $f : X \to Y$ is an **isomorphism** if there exists a morphism $g : Y \to X$ (called the inverse) such that:

$$g \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g = \mathrm{id}_Y .$$

If such an f exists, we say X and Y are **isomorphic** ($X \cong Y$).

Note 11 Bimorphisms are not Isomorphisms.

A morphism that is both mono and epi is called a **bimorphism**. **Warning:** A bimorphism is NOT necessarily an isomorphism!

i. In Set and Ab, Bimorphism $\iff$ Isomorphism.

ii. In Ring, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is both mono and epi, but clearly not an isomorphism.

iii. In Top, a continuous bijection is both mono and epi, but not necessarily a homeomorphism (the inverse might not be continuous).

In algebraic geometry, checking isomorphism requires constructing the inverse explicitly.

Limits and colimits generalize constructions like products, intersections, kernels, and direct sums. They are defined via **Universal Properties**.

Definition 12 Product.

Let $\{X_i\}_{i \in I}$ be a family of objects in C. A **product** is an object P (denoted $\prod X_i$) together with projection morphisms $\pi_i : P \to X_i$, satisfying the following universal property: For any object Y and any family of morphisms $f_i : Y \to X_i$, there exists a **unique** morphism $f : Y \to P$ such that $\pi_i \circ f = f_i$ for all i.

$$\begin{tikzcd}
Y \arrow[d, dashed, "f"] \arrow[dr, "f_i"] & \\
\prod X_i \arrow[r, "\pi_i"] & X_i
\end{tikzcd}$$

Definition 13 Coproduct / Direct Sum.

The **coproduct** is the dual notion. A coproduct of $\{X_i\}$ is an object C (denoted $\coprod X_i$ or $\bigoplus X_i$) together with injection morphisms $\iota_i : X_i \to C$, satisfying: For any object Y and morphisms $g_i : X_i \to Y$, there exists a **unique** morphism $g : C \to Y$ such that $g \circ \iota_i = g_i$.

Let $\mathscr{I}$ be a "small" index category and $F : \mathscr{I} \to C$ a functor (a diagram).

Definition 14 Inverse Limit.

The **inverse limit** (or projective limit, simply limit), denoted $\varprojlim F$ or $\lim F$, is an object L together with morphisms $\pi_i : L \to F(i)$ for each $i \in \mathscr{I}$, such that:

1. For every arrow $u : i \to j$ in $\mathscr{I}$, we have $F(u) \circ \pi_i = \pi_j$. (Compatibility)
2. (Universality) For any other object Y with compatible maps to $F(i)$, there is a unique map $Y \to L$.

Examples: Kernels, Products, Intersections.

Definition 15 Direct Limit.

The **direct limit** (or inductive limit, colimit), denoted $\varinjlim F$ or $\operatorname{colim} F$, is the dual notion. It is an object C with maps $\iota_i : F(i) \to C$ such that:

1. For $u : i \to j$, we have $\iota_j \circ F(u) = \iota_i$.
2. (Universality) For any Y with compatible maps from $F(i)$, there is a unique map $C \to Y$.

Examples: Cokernels, Coproducts, Quotients, Stalks.

Note 16 Limits and Colimits in Sheaf Theory.

In Scheme Theory, we constantly switch between these two:

i. **Sheaf Axiom = Limit:** The condition for a presheaf to be a sheaf is that $\mathscr{F}(U)$ must be the **equalizer** (a limit) of the restriction maps on a cover.

ii. **Stalk = Colimit:** The stalk $\mathscr{F}_x$ is the **direct limit** of sections over neighborhoods $U \ni x$.

Understanding the commutativity (or lack thereof) between limits and colimits is key to cohomology.

To do homological algebra (kernels, exact sequences, cohomology), we need categories with more structure than just sets and maps. We build this structure in layers: from Additive to Abelian.

Definition 17 Zero Object.

An object 0 is a **zero object** if it is both initial and terminal. That is, for any object X, there are unique morphisms $0 \to X$ and $X \to 0$. If a category has a zero object, we define the **zero morphism** $0_{XY} : X \to Y$ as the unique composite $X \to 0 \to Y$.

Definition 18 Additive Category.

A category C is **additive** if:

1. It has a zero object.

2. It has all finite binary products (and thus coproducts). In fact, in this setting, products and coproducts coincide: $X \times Y \cong X \oplus Y$ (called **biproducts**).
3. Every set of morphisms $\mathrm{Hom}(X, Y)$ has the structure of an abelian group, such that composition is bilinear.

Examples: Ab, Mod$_R$, but **not** Ring or Set.

Let $f : X \to Y$ be a morphism in an additive category (so we can talk about the zero morphism).

Definition 19 Kernel and Cokernel.

i. The **kernel** of f, denoted $\ker f$, is the equalizer of f and the zero morphism 0_{XY}. Specifically, it is a morphism $k : K \to X$ such that $f \circ k = 0$, satisfying the universal property: for any $g : Z \to X$ with $f \circ g = 0$, g factors uniquely through k.

ii. The **cokernel** of f, denoted $\operatorname{coker} f$, is the coequalizer of f and 0_{XY}. It is a morphism $c : Y \to C$ such that $c \circ f = 0$, satisfying the dual universal property.

In nice categories, every morphism induces a decomposition of the objects. We define the Image and Coimage **using** Kernels and Cokernels. This is the categorical way to avoid talking about "elements".

Definition 20 Image and Coimage.

i. The **image** of f, denoted $\operatorname{im} f$, is the kernel of the cokernel:

$$\operatorname{im} f := \ker(\operatorname{coker} f).$$

(Think: The "image" is the subspace of Y that maps to 0 in the quotient $Y/f(X)$).

ii. The **coimage** of f, denoted $\operatorname{coim} f$, is the cokernel of the kernel:

$$\operatorname{coim} f := \operatorname{coker}(\ker f).$$

(Think: The "coimage" is the quotient space $X/\ker f$).

An additive category is **Abelian** if it behaves like the category of abelian groups. The critical requirement is the First Isomorphism Theorem.

Definition 21 Abelian Category.

An additive category $\mathcal{A}$ is **Abelian** if:

1. Every morphism has a kernel and a cokernel.
2. Every monomorphism is the kernel of its cokernel.
3. Every epimorphism is the cokernel of its kernel.
4. **(The Isomorphism Theorem)** For every morphism $f : X \to Y$, the canonical induced morphism $\bar{f} : \operatorname{coim} f \to \operatorname{im} f$ is an **isomorphism**.

Note 22 Canonical Factorization.

In an Abelian category, every morphism $f : X \to Y$ admits a unique canonical factorization:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, two heads, "\mathrm{coim}"'] & & Y \\
& I \arrow[ur, hook, "\mathrm{im}"'] &
\end{tikzcd}$$

where $X \to I$ is an epimorphism (the coimage) and $I \to Y$ is a monomorphism (the image). Because $\mathrm{coim}\, f \cong \mathrm{im}\, f$, we identify them and call I "the image".

With kernels and images defined via universal properties, we can rigorously define exact sequences in any Abelian category (including sheaves).

Definition 23 Exact Sequence.

A sequence of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ is **exact at** Y if:

$$\mathrm{im}\, f = \ker g.$$

(More precisely, the monomorphism $\mathrm{im}\, f \hookrightarrow Y$ is equivalent to the monomorphism $\ker g \hookrightarrow Y$). A sequence $0 \to X \to Y \to Z \to 0$ is a **short exact sequence** if it is exact at X, Y, and Z. This is equivalent to saying $X = \ker(Y \to Z)$ and $Z = \mathrm{coker}(X \to Y)$.

Note 24 Why do we care?.

The category of sheaves of abelian groups on a space X, denoted $\mathsf{Sh}(X)$, is an **Abelian Category**. This is a non-trivial fact! It relies on the sheafification functor to construct cokernels. Because $\mathsf{Sh}(X)$ is Abelian, we can perform all standard homological algebra operations (Snake Lemma, Five Lemma, Long Exact Sequence of Cohomology) directly on sheaves.

"Adjoint functors arise everywhere." This slogan captures the essence of modern mathematics. Most canonical constructions (free objects, completions, spectra, sheafification) are instances of adjoint functors.

The most intuitive definition of adjunction is through a natural bijection of Hom-sets. It expresses a duality between two functors.

Definition 25 Adjunction.

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors. We say that F is the **left adjoint** to G (and G is the **right adjoint** to F), denoted $F \dashv G$, if there exists a bijection of sets:

$$\Phi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

which is **natural** in both $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.

Naturality means that for any morphisms $f : X' \to X$ in $\mathcal{C}$ and $g : Y \to Y'$ in $\mathcal{D}$, the following diagrams commute (compatible with pre-composition and post-composition).

By analyzing the identity morphisms under the bijection Φ, we extract two fundamental natural transformations that characterize the adjunction.

Definition 26 Unit and Counit of an Adjunction.

Let $F \dashv G$.

i. **Unit:** By setting $Y = F(X)$, the identity $\mathrm{id}_{F(X)}$ corresponds to a morphism $\eta_X : X \to G(F(X))$. This defines a natural transformation:

$$\eta : \mathrm{id}_{\mathcal{C}} \longrightarrow G \circ F$$

called the **unit** of the adjunction.

ii. **Counit:** By setting $X = G(Y)$, the identity $\mathrm{id}_{G(Y)}$ corresponds to a morphism $\varepsilon_Y : F(G(Y)) \to Y$. This defines a natural transformation:

$$\varepsilon : F \circ G \longrightarrow \mathrm{id}_{\mathcal{D}}$$

called the **counit** of the adjunction.

Theorem 27 Triangle Identities.

Two functors F, G form an adjunction $(F, G, \eta, \varepsilon)$ if and only if they satisfy the **Triangle Identities**:

1. The composition $F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F$ is the identity id_F.
2. The composition $G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$ is the identity id_G.

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "\mathrm{id}_F"'] & FGF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "\mathrm{id}_G"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd}$$

This is the most practically useful theorem for algebraic geometry. It tells us "for free" whether a functor is exact (or half-exact).

Theorem 28 RAPL and LAPC.

Let $F : \mathcal{C} \to \mathcal{D}$ be a left adjoint and $G : \mathcal{D} \to \mathcal{C}$ be a right adjoint ($F \dashv$ G).

1. **Right Adjoints Preserve Limits (RAPL):** G preserves all limits that exist in $\mathcal{D}$.

$$G(\varprojlim Y_i) \cong \varprojlim G(Y_i).$$

In particular, G preserves kernels and products. Thus, a right adjoint between abelian categories is always **left exact**.

2. **Left Adjoints Preserve Colimits (LAPC):** F preserves all colimits that exist in $\mathcal{C}$.

$$F(\varinjlim X_i) \cong \varinjlim F(X_i).$$

In particular, F preserves cokernels and direct sums. Thus, a left adjoint between abelian categories is always **right exact**.

We summarize the key adjunctions used in this book.

Example 29 Module Operations.

Let $f : X \to Y$ be a morphism of ringed spaces.

i. The inverse image f^* is left adjoint to the direct image f_*.

$$\mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{G}, f_*\mathscr{F}).$$

Consequence: f^* is right exact (preserves cokernels/tensor products), while f_* is left exact (preserves kernels). This explains why we need derived functors ($R^i f_*$) for direct images but not usually for inverse images.

Example 30 Sheafification.

Let $\mathsf{PSh}(X)$ be the category of presheaves and $\mathsf{Sh}(X)$ the category of sheaves. The sheafification functor $(-)^+ : \mathsf{PSh}(X) \to \mathsf{Sh}(X)$ is left adjoint to the inclusion (forgetful) functor ι.

$$\mathrm{Hom}_{\mathsf{Sh}}(\mathscr{F}^+, \mathscr{G}) \cong \mathrm{Hom}_{\mathsf{PSh}}(\mathscr{F}, \iota\mathscr{G}).$$

Consequence: Sheafification preserves colimits (like stalks and cokernels), which is why $(\mathscr{F}^+)_x \cong \mathscr{F}_x$.

Example 31 Free Modules.

Let A be a ring. The "free module functor" $S \mapsto A^{\oplus S}$ (from Sets to A-Modules) is left adjoint to the forgetful functor (taking the underlying set).

$$\mathrm{Hom}_A(A^{\oplus S}, M) \cong \mathrm{Hom}_{\mathsf{Set}}(S, M).$$

1.2 Fundamental Lemmas of Homological Algebra

Proving diagram lemmas (like the Snake Lemma or Five Lemma) using only the universal properties of kernels and cokernels in an arbitrary abelian category is notoriously tedious. However, we can bypass this complexity using a powerful "metatheorem" from category theory.

Theorem 32 Freyd-Mitchell Embedding Theorem.

Every small abelian category admits an exact, fully faithful embedding into the category of modules over some ring R.

To prove a diagrammatic statement about exactness in any abelian category, it suffices to prove it for categories of modules. This allows us to use the method of **"diagram chasing"** with elements, which is far more intuitive. In the proofs below, we treat objects as modules and morphisms as linear maps.

Lemma 33 The Snake Lemma.

Consider a commutative diagram with exact rows:

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d, "u"] & B \arrow[r] \arrow[d, "v"] & C \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' &
\end{tikzcd}$$

Then there exists a connecting homomorphism $\delta : \ker w \to \operatorname{coker} u$ such that the following sequence is exact:

$$\ker u \to \ker v \to \ker w \xrightarrow{\delta} \operatorname{coker} u \to \operatorname{coker} v \to \operatorname{coker} w.$$

Proof. The exactness at $\ker u, \ker v$ and $\operatorname{coker} v, \operatorname{coker} w$ is standard. The non-trivial part is the construction of δ and exactness at the "snake" turn.

Step 1: Construction of δ (The Switchback): Let $z \in \ker w \subseteq C$.

Since $B \to C$ is surjective, lift z to $y \in B$. Map y down to B' to get $v(y)$. Commutativity implies the image of $v(y)$ in C' is $w(z) = 0$. By exactness of the bottom row, $v(y)$ comes from a unique element $x' \in A'$. Define $\delta(z) := [x'] \in \operatorname{coker} u = A'/\operatorname{im}(u)$.

One checks easily that δ is well-defined (independent of the choice of lift y).

Step 2: Exactness at $\ker w$: Let $z \in \ker w$. $\delta(z) = 0 \iff x' \in \operatorname{im}(u) \iff x' = u(x)$ for some $x \in A$. By injectivity of $A' \to B'$, $v(y) =$ image of $u(x)$. By commutativity, $v(y) = v(\text{image of } x)$. Thus $y - (\text{image of } x) \in \ker v$. This means z (which is the image of y) is the image of an element in $\ker v$.

Step 3: Exactness at $\operatorname{coker} u$: Let $\bar{x'} \in \operatorname{coker} u$ be the class of $x' \in A'$. Maps to 0 in $\operatorname{coker} v \iff$ the image of x' in B' is in $\operatorname{im}(v)$. Let this image be $v(y)$ for some $y \in B$. Let z be the image of y in C. Then $w(z) =$ image of $v(y) =$ image of (image of x') $= 0$. So $z \in \ker w$. By construction, $\delta(z) = \bar{x'}$. □

Instead of chasing the massive diagram of the Five Lemma directly, we deduce it from the **Short Five Lemma**, which is cleaner.

Lemma 34 Short Five Lemma.

Consider a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

i. If α, γ are injective, then β is injective.

ii. If α, γ are surjective, then β is surjective.

iii. If α, γ are isomorphisms, then β is an isomorphism.

Proof. (i) Suppose $\beta(b) = 0$. The image in C' is $\gamma(\text{image of } b) = 0$. Since γ is injective, the image of b in C is 0. By exactness, b comes from $a \in A$. Then $\alpha(a)$ maps to $\beta(b) = 0$ in B'. Since $A' \to B'$ is injective, $\alpha(a) = 0$. Since α is injective, $a = 0$, so $b = 0$. (ii) Similar diagram chasing (or duality). (iii) follows from (i) and (ii). □

Theorem 35 The Five Lemma.

onsider the commutative diagram with exact rows:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[r] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

If f_1, f_2, f_4, f_5 are isomorphisms, then f_3 is an isomorphism.

Proof. We break the diagram into two "short" pieces. Let $K_i = \ker(A_i \to A_{i+1}) = \operatorname{im}(A_{i-1} \to A_i)$ (and similarly L_i for B's). The diagram induces maps on these kernels/images.

1. **Right side:** The map f_4 restricts to an isomorphism between $\operatorname{im}(A_3 \to A_4)$ and $\operatorname{im}(B_3 \to B_4)$ (injectivity uses f_5, surjectivity uses f_4). Let's call these images I_3 and J_3.
2. **Left side:** Similarly, f_2 induces an isomorphism between $\operatorname{coker}(A_1 \to A_2)$ and $\operatorname{coker}(B_1 \to B_2)$. Let's call these K_3 and L_3.
3. **Middle:** We now have a short exact sequence diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_3 \arrow[r] \arrow[d, "\cong"] & A_3 \arrow[r] \arrow[d, "f_3"] & I_3 \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & L_3 \arrow[r] & B_3 \arrow[r] & J_3 \arrow[r] & 0
\end{tikzcd}$$

   By the Short Five Lemma, f_3 is an isomorphism.

□

CHAPTER 2

Presheaves and Sheaves

2.1 Presheaves

Intuition 1 Intuition behind Sheaves.

The concept of a sheaf is the fundamental tool for organizing local data over a topological space. Rather than defining it through a long list of axioms involving restriction maps, we adopt the language of category theory. This perspective reveals that a presheaf is simply a contravariant functor, and a morphism of presheaves is a natural transformation.

To treat presheaves as functors, we must first view the topology of a space as a category.

Definition 2 The Category $\mathsf{Op}(X)$.

Let X be a topological space. We define the category $\mathsf{Op}(X)$ as follows:

i. **Objects:** The open subsets $U \subseteq X$.

ii. **Morphisms:** For any two open sets U, V, the set of morphisms is defined by inclusion:

$$\mathrm{Hom}_{\mathsf{Op}(X)}(V, U) = \begin{cases} \{\iota_V^U\} & \text{if } V \subseteq U, \\ \emptyset & \text{otherwise}, \end{cases}$$

where $\iota_V^U : V \hookrightarrow U$ denotes the inclusion map.

Composition of morphisms corresponds to the transitivity of inclusion ($W \subseteq V \subseteq U$).

Let C be a target category. In algebraic geometry, C is typically the category of abelian groups (Ab), rings (Ring), or sets (Set).

Definition 3 Presheaf.

A **presheaf** $\mathscr{F}$ on X with values in C is a contravariant functor from $\mathsf{Op}(X)$ to C.

$$\mathscr{F} : \mathsf{Op}(X)^{\mathrm{op}} \longrightarrow C.$$

Let us translate the functorial definition into classical terms to see that it recovers the familiar notion.

1. **Objects (Sections):** For every open set U, the functor assigns an object $\mathscr{F}(U)$ in $\mathcal{C}$. Elements of $\mathscr{F}(U)$ are called **sections** over U, often denoted by $\Gamma(U, \mathscr{F})$.

2. **Morphisms (Restriction):** For every inclusion $V \hookrightarrow U$ (a morphism in $\mathsf{Op}(X)$), the contravariant functor assigns a morphism in $\mathcal{C}$:

$$\rho_{UV} : \mathscr{F}(U) \longrightarrow \mathscr{F}(V).$$

   We usually write $\rho_{UV}(s) = s|_V$ for a section $s \in \mathscr{F}(U)$.

3. **Functoriality (Axioms):**

   i. **Identity:** The functor preserves identity morphisms. The inclusion $U \subseteq U$ maps to the identity map $\mathrm{id} : \mathscr{F}(U) \to \mathscr{F}(U)$. Thus $s|_U = s$.

   ii. **Composition:** The functor preserves composition. If $W \subseteq V \subseteq U$, then the diagram commutes:

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\rho_{UV}"] \arrow[rr, bend right, "\rho_{UW}"'] & \mathscr{F}(V) \arrow[r, "\rho_{VW}"] & \mathscr{F}(W)
\end{tikzcd}$$

   This means $(s|_V)|_W = s|_W$.

Since presheaves are functors, the appropriate notion of a morphism between them is provided by category theory.

Definition 4 Morphism of Presheaves.

Let $\mathscr{F}$ and $\mathscr{G}$ be presheaves on X with values in $\mathcal{C}$. A **morphism** $\phi : \mathscr{F} \to \mathscr{G}$ is a **natural transformation** between the functors.

Explicitly, ϕ consists of a collection of morphisms $\{\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)\}$ for each open U, such that for every inclusion $V \subseteq U$, the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\phi(U)"] \arrow[d, "\mathrm{res}"] & \mathscr{G}(U) \arrow[d, "\mathrm{res}"] \\
\mathscr{F}(V) \arrow[r, "\phi(V)"] & \mathscr{G}(V)
\end{tikzcd}$$

This diagram expresses the compatibility of the morphism with restriction maps: $\phi(U)(s)|_V = \phi(V)(s|_V)$.

Note 5 The Category of Presheaves.

The presheaves on X with values in $\mathcal{C}$ form a category, denoted $\mathsf{PSh}(X, \mathcal{C})$ or simply $\mathsf{PSh}(X)$. Since limits and colimits in functor categories are computed pointwise, if $\mathcal{C}$ is an abelian category (like Ab), then $\mathsf{PSh}(X)$ is also an **abelian category**. This means we can talk about kernels, cokernels, and exact sequences of presheaves simply by looking at them open set by open set. For example:

$$(\ker \phi)(U) = \ker(\phi(U)).$$

Warning: As we shall see, the category of **sheaves** is also abelian, but kernels and cokernels behave differently there (specifically cokernels).

To build intuition, we examine several concrete examples. We interpret them through our functorial definition: $\mathscr{F} : \mathsf{Op}(X)^{\mathrm{op}} \to C$.

Example 6 The Presheaf of Functions.

This is the prototypical example. Let X be a topological space (e.g., a manifold or a complex domain). Let C be the category of commutative rings. Define the functor C^0_X by:

i. **On Objects:** For any open set U, $C^0_X(U)$ is the ring of continuous complex-valued functions on U.

ii. **On Morphisms:** For an inclusion $V \hookrightarrow U$, the map $C^0_X(U) \to C^0_X(V)$ is the classical **restriction of functions**:

$$f \longmapsto f|_V.$$

Functoriality is satisfied because $(f|_V)|_W = f|_W$ and $f|_U = f$. Variations of this example include:

i. C^∞_X: The presheaf of smooth functions (on a smooth manifold).

ii. $\mathscr{O}_{hol}$: The presheaf of holomorphic functions (on a complex manifold).

Example 7 The Constant Presheaf.

Let A be an abelian group (or a ring). We can define a "constant functor" $\underline{A}^{\mathrm{pre}}$:

i. For every non-empty open set U, $\underline{A}^{\mathrm{pre}}(U) = A$.

ii. For every inclusion $V \subseteq U$ of non-empty sets, the restriction map is the identity id_A.

iii. (Conventionally, we set $\underline{A}^{\mathrm{pre}}(\emptyset) = 0$).

This is called the **constant presheaf** associated to A. **Warning:** As we shall see later, this presheaf is usually **not** a sheaf (unless X is irreducible), because global constant sections cannot represent "locally constant but globally different" functions on disjoint open sets.

Example 8 Sections of a Continuous Map.

This example generalizes the concept of functions. Let $\pi : E \to X$ be a continuous map of topological spaces. (One may think of E as a vector bundle or a covering space over X). We define the presheaf of **sections**, denoted by $\mathscr{S}_\pi$, as follows:

i. **On Objects:** For an open set U, $\mathscr{S}_\pi(U)$ is the set of continuous sections of π over U:

$$\mathscr{S}_\pi(U) := \{s : U \to E \mid \pi \circ s = \mathrm{id}_U,\ s \text{ is continuous}\}.$$

ii. **On Morphisms:** For $V \subseteq U$, the restriction map is the standard restriction of the map s.

Note that Example 1 is a special case of this, where $E = X \times \mathbb{C}$ is the trivial bundle and π is the projection.

Example 9 A Pathological Example: Bounded Functions.

Let $X = \mathbb{R}$. Define $\mathcal{B}(U)$ to be the set of **bounded** continuous functions on U. Restriction of a bounded function to a smaller set is still bounded, so this defines a valid presheaf. However, this example highlights the limitation of presheaves. Being "bounded" is not a local property: the function $f(x) = x$ is locally bounded everywhere (bounded on any compact interval), but it is not a section in $\mathcal{B}(\mathbb{R})$. Thus, this presheaf fails the "gluing axiom" which we will introduce in the next section.

2.2 Sheaves

A presheaf provides data over open sets, but it does not guarantee that this data behaves geometrically. The concept of a **sheaf** enforces the "local-to-global" principle: global data is precisely determined by compatible local data.

Let X be a topological space and let $\mathscr{F}$ be a presheaf on X. Let $U \subseteq X$ be an open set and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of U (i.e., $U = \bigcup_{i\in I} U_i$).

We can form a sequence of maps based on the restrictions:

$$\mathscr{F}(U) \xrightarrow{\alpha} \prod_{i\in I} \mathscr{F}(U_i) \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathscr{F}(U_i \cap U_j)$$

where:

i. The map α sends a section $s \in \mathscr{F}(U)$ to the family of its restrictions: $\alpha(s) = (s|_{U_i})_{i\in I}$.

ii. The maps β and γ act on a family $(s_i)_{i\in I}$ by restricting to the overlaps:

$$\beta((s_i)) = (s_i|_{U_i\cap U_j})_{i,j} \quad \text{and} \quad \gamma((s_i)) = (s_j|_{U_i\cap U_j})_{i,j}.$$

Definition 10 Sheaf.

A presheaf $\mathscr{F}$ is a **sheaf** if for *every* open set U and *every* open covering $\{U_i\}$ of U, the object $\mathscr{F}(U)$ is the **equalizer** of the maps β and γ.

In the category of abelian groups (or rings, modules), the condition that $\mathscr{F}(U)$ is the equalizer is equivalent to the exactness of the following sequence:

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\alpha} \prod_{i\in I} \mathscr{F}(U_i) \xrightarrow{\delta} \prod_{i,j\in I} \mathscr{F}(U_i \cap U_j)$$

where $\delta = \beta - \gamma$, i.e., it maps (s_i) to $(s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})$.

The exactness of the sequence above encodes two fundamental geometric axioms:

1. **Injectivity (Monopresheaf / Separatedness):** The map α is injective.

   Translation: If $s, t \in \mathscr{F}(U)$ satisfy $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$. (Local equality implies global equality).

2. **Exactness in the Middle (Gluing / Existence):** The image of α is the kernel of δ.

   Translation: Given a family of local sections $s_i \in \mathscr{F}(U_i)$ that are compatible on overlaps ($s_i|_{U_{ij}} = s_j|_{U_{ij}}$), there exists a global section $s \in \mathscr{F}(U)$ that restricts to each s_i.

While the definition above is elegant, checking it in practice is daunting. The definition requires the axioms to hold for **arbitrary** open sets U and **arbitrary** coverings. In algebraic geometry, we prioritize specific open sets—namely, the principal open sets $D(f)$ which form a basis for the topology.

This raises a natural question: **Is it sufficient to verify the sheaf axioms only for a basis of the topology?**

The answer is affirmative and leads to the **Extension Theorem**, which we discuss next. This theorem allows us to construct sheaves (like the structure sheaf of a scheme) by defining them only on affine open sets and extending uniquely.

Definition 11 Basis of Topology.

Let X be a topological space. A collection of open sets $\mathcal{B}$ is a **basis** if:

1. $\mathcal{B}$ covers X.
2. For any $U, V \in \mathcal{B}$ and $x \in U \cap V$, there exists $W \in \mathcal{B}$ such that $x \in W \subseteq U \cap V$.

(Note: In our applications, the intersection of two basic sets is often itself a basic set, e.g., $D(f) \cap D(g) = D(fg)$, which simplifies things further.)

Definition 12 Sheaf on a Basis / $\mathcal{B}$-Sheaf.

Let $\mathcal{B}$ be a basis for X. A **$\mathcal{B}$-sheaf** $\mathscr{F}$ consists of:

1. For every $U \in \mathcal{B}$, an abelian group (or ring/module) $\mathscr{F}(U)$.
2. For every inclusion $V \subseteq U$ with $U, V \in \mathcal{B}$, a restriction map $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$, satisfying the usual presheaf axioms (identity and composition).
3. **Gluing Axiom:** For any $U \in \mathcal{B}$ and any covering $\{V_i\}_{i\in I}$ of U by basic sets $V_i \in \mathcal{B}$, the sequence
$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\alpha} \prod_{i\in I} \mathscr{F}(V_i) \xrightarrow{\beta} \prod_{i,j\in I} \mathscr{F}(V_i \cap V_j)$$
is exact. (Here, the rightmost term implies checking equality on a basic covering of $V_i \cap V_j$).

Theorem 13 Extension Theorem for Sheaves on a Basis.

Let $\mathcal{B}$ be a basis for X and $\mathscr{F}$ be a $\mathcal{B}$-sheaf. Then:

1. There exists a sheaf $\mathscr{F}_{ext}$ on X extending $\mathscr{F}$, meaning $\mathscr{F}_{ext}(U) \cong \mathscr{F}(U)$ naturally for all $U \in \mathcal{B}$.
2. This extension is unique up to unique isomorphism.
3. The stalks of $\mathscr{F}_{ext}$ are isomorphic to the stalks computed using the basis limits.

Proof. Let $\mathcal{B}$ be a basis for the topology of X and let $\mathscr{F}$ be a $\mathcal{B}$-sheaf.

Step 1: Construction via Inverse Limits. For an arbitrary open set $U \subseteq X$, we define $\mathscr{F}_{ext}(U)$ not by a random assignment, but by the "limit" of all information available on the basis subsets.

$$\mathscr{F}_{ext}(U) := \varprojlim_{V \in \mathcal{B}, V \subseteq U} \mathscr{F}(V).$$

Explicitly, an element $s \in \mathscr{F}_{ext}(U)$ is a collection of sections $s = (s_V)_{V \subseteq U, V \in \mathcal{B}}$ such that for any inclusion $W \subseteq V$ of basis sets inside U, the restriction holds: $s_V|_W = s_W$. The restriction maps for $\mathscr{F}_{ext}$ are defined naturally by restricting the family of subsets. This clearly defines a presheaf.

Step 2: Verification of the Sheaf Axioms. We must show that $\mathscr{F}_{ext}$ is a sheaf. Let $U = \bigcup_{i \in I} U_i$ be an open covering.

i. **Injectivity (Separatedness):** Suppose $s \in \mathscr{F}_{ext}(U)$ restricts to 0 on each U_i. This means for every i, the family defining s vanishes on all basis sets contained in U_i. For *any* basis set $V \subseteq U$, we have $V = \bigcup_i (V \cap U_i)$. Since $\mathcal{B}$ is a basis, each $V \cap U_i$ is covered by basis sets $\{W_{ij}\}$. Since s vanishes on U_i, it vanishes on W_{ij}. Because $\mathscr{F}$ is a $\mathcal{B}$-sheaf (satisfying the sheaf axiom on the basis), the fact that s_V vanishes on a basic cover $\{W_{ij}\}$ of V implies $s_V = 0$. Thus $s = 0$.

ii. **Surjectivity (Gluing):** Let $s^{(i)} \in \mathscr{F}_{ext}(U_i)$ be a compatible family. We want to construct a global s. For any basis set $V \subseteq U$, the family $\{V \cap U_i\}$ covers V. Again, refine this to a basis cover $\{W_{ij}\}$ of V, where $W_{ij} \subseteq U_i$. On each W_{ij}, we have a section provided by $s^{(i)}$. By the compatibility of $s^{(i)}$ and $s^{(j)}$, these agree on overlaps. Since $\mathscr{F}$ is a $\mathcal{B}$-sheaf, these sections glue uniquely to a section $s_V \in \mathscr{F}(V)$. The collection (s_V) forms the required global section s.

Step 3: Compatibility with the Basis. For $U \in \mathcal{B}$, is $\mathscr{F}_{ext}(U) \cong \mathscr{F}(U)$? Yes. The element U is the terminal object in the system of basis sets contained in U. The inverse limit over a system with a terminal object is simply the value at that object. Thus $\mathscr{F}_{ext}(U) \cong \mathscr{F}(U)$.

Step 4: Uniqueness. Any sheaf $\mathscr{G}$ is determined by its values on a basis, because for any U, $\mathscr{G}(U)$ must be the equalizer of the product of its restrictions to a basic cover. This forces $\mathscr{G}(U) \cong \varprojlim \mathscr{G}(V)$ for $V \in \mathcal{B}$. □

We now revisit our examples of presheaves. Some are already sheaves, while others fail the axioms and require "sheafification" (or the sheaf extension process described above).

Example 14 Continuous Functions: A Sheaf.

The presheaf of continuous functions C_X^0 is indeed a **sheaf**.

i. **Locality:** If a function is zero in a neighborhood of every point, it is the zero function.

ii. **Gluing:** If we define continuous functions f_i on U_i that agree on overlaps, their set-theoretic union f is a well-defined function. By the local nature of continuity (inverse image of open is open), f is continuous on $\bigcup U_i$.

This applies to C^∞, holomorphic functions, and regular functions in algebraic geometry.

Example 15 Disconnected Space Example.

Let X be a disconnected space, say $X = U \sqcup V$ (disjoint open sets). Let $A = \mathbb{Z}$.

i. **The Presheaf:** The constant presheaf $\underline{\mathbb{Z}}^{pre}$ assigns $\mathbb{Z}$ to every non-empty open set. Consider sections $s_U = 1 \in \mathbb{Z}$ on U and $s_V = 2 \in \mathbb{Z}$ on V. Their intersection is empty, so the compatibility condition is vacuously satisfied. However, is there a global section $s \in \underline{\mathbb{Z}}^{pre}(X) = \mathbb{Z}$ that restricts to 1 on U and 2 on V? **Impossible.** Thus, the constant presheaf is **not** a sheaf.

ii. **The Sheaf:** The sheaf associated to this presheaf is the **locally constant sheaf**, denoted $\underline{\mathbb{Z}}_X$.
$$\underline{\mathbb{Z}}_X(W) = \{f : W \to \mathbb{Z} \mid f \text{ is locally constant}\}.$$
In the case above, $\underline{\mathbb{Z}}_X(U \sqcup V) \cong \mathbb{Z} \oplus \mathbb{Z}$. The section $(1, 2)$ is a valid global section.

Example 16 Bounded Functions: Sheafification.

Recall the presheaf of bounded functions $\mathcal{B}$. Consider $X = \mathbb{R}$ and the cover $X = \bigcup_n (-n, n)$. The function $f(x) = x$ is bounded on every element of the cover (it is bounded on compact sets). However, $f(x)$ is **not** bounded on $\mathbb{R}$. Thus, the presheaf $\mathcal{B}$ fails the gluing axiom (we can glue the local restrictions, but the result falls out of the "bounded" category). The sheafification of $\mathcal{B}$ is the sheaf of **locally bounded functions**. On a locally compact space like $\mathbb{R}$, this is simply the sheaf of all continuous functions.

While sheaves encode data over open sets, we often need to focus on the behavior of the data near a single point. This leads to the concept of the stalk.

Definition 17 Stalk.

Let $\mathcal{F}$ be a presheaf on X and let $x \in X$. The **stalk** of $\mathcal{F}$ at x, denoted $\mathcal{F}_x$, is the direct limit (colimit) of the sections over all open neighborhoods of x:
$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U).$$

Elements of $\mathcal{F}_x$ are called **germs**. A germ is represented by a pair (U, s) where $x \in U$ and $s \in \mathcal{F}(U)$. Two pairs (U, s) and (V, t) define the same germ in $\mathcal{F}_x$ if there exists a smaller neighborhood $W \subseteq U \cap V$ containing x such that $s|_W = t|_W$. We denote the image of a section $s \in \mathcal{F}(U)$ in the stalk by s_x.

Note 18 Exactness of Stalks.

The operation of taking the stalk, $\mathcal{F} \mapsto \mathcal{F}_x$, is a functor $\mathsf{PSh}(X) \to C$ (e.g., Ab). Crucially, direct limits preserve exactness. Thus, taking stalks is an **exact functor**. This means we can check exactness of a sequence of *sheaves* simply by checking it at every stalk.

One of the most powerful features of sheaf theory is that many global properties can be checked point-by-point on stalks.

Proposition 19 Isomorphism via Stalks.

Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of **sheaves** on a topological space X. Then ϕ is an isomorphism if and only if for every point $x \in X$, the induced map on stalks $\phi_x : \mathscr{F}_x \to \mathscr{G}_x$ is an isomorphism.

Proof. ($\Rightarrow$) Since taking stalks is a functor (specifically, a directed colimit), it preserves isomorphisms. If ϕ has an inverse ψ, then $(\phi_x)^{-1} = (\psi)_x$.

($\Leftarrow$) Suppose ϕ_x is an isomorphism for all x. We must show that for any open set U, the map $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is bijective.

Step 1: Injectivity. Let $s \in \mathscr{F}(U)$ be a section such that $\phi_U(s) = 0$ in $\mathscr{G}(U)$. For any $x \in U$, the image in the stalk is $(\phi_U(s))_x = \phi_x(s_x) = 0$. Since ϕ_x is injective, we have $s_x = 0$ in $\mathscr{F}_x$. By the definition of the stalk, $s_x = 0$ implies there exists an open neighborhood $V_x \subseteq U$ of x such that $s|_{V_x} = 0$. The family $\{V_x\}_{x \in U}$ covers U. Since $\mathscr{F}$ is a sheaf (specifically, it satisfies the locality/monopresheaf axiom), a section that is zero locally everywhere is zero globally. Thus $s = 0$.

Step 2: Surjectivity. Let $t \in \mathscr{G}(U)$. We want to find $s \in \mathscr{F}(U)$ such that $\phi_U(s) = t$. For each $x \in U$, consider the germ $t_x \in \mathscr{G}_x$. Since ϕ_x is surjective, there exists a germ $r_x \in \mathscr{F}_x$ such that $\phi_x(r_x) = t_x$. By definition of the stalk, the germ r_x is represented by a section $s^{(x)}$ defined on some neighborhood $V_x \subseteq U$ of x. So $(s^{(x)})_x = r_x$.

Issue: We have a collection of local candidates $s^{(x)}$ on V_x. Do they map to t? $\phi(s^{(x)})$ and t have the same germ at x (by construction). Thus, they must agree on some smaller neighborhood $W_x \subseteq V_x$. Let's replace V_x with W_x and $s^{(x)}$ with its restriction. Now we have $\phi(s^{(x)}) = t|_{W_x}$.

Gluing: We have a cover $U = \bigcup_{x \in U} W_x$ and sections $s^{(x)} \in \mathscr{F}(W_x)$. Do they glue? Consider the intersection $W_x \cap W_y$. On this overlap, we have two sections $s^{(x)}$ and $s^{(y)}$. Their images under ϕ are $t|_{W_x \cap W_y}$ and $t|_{W_x \cap W_y}$, which are identical. Thus $\phi(s^{(x)} - s^{(y)}) = 0$ on the overlap. Since we have already proven that ϕ is **injective** (Step 1 applies to sheaves on any open set), this implies $s^{(x)} - s^{(y)} = 0$, i.e., $s^{(x)} = s^{(y)}$ on the overlap. Since $\mathscr{F}$ is a sheaf, these compatible local sections glue to a unique global section $s \in \mathscr{F}(U)$. Finally, $\phi(s)|_{W_x} = \phi(s|_{W_x}) = \phi(s^{(x)}) = t|_{W_x}$. Since $\mathscr{G}$ is a sheaf, $\phi(s) = t$. □

A presheaf $\mathscr{F}$ may fail to be a sheaf. We wish to associate to it the "best possible" sheaf $\mathscr{F}^+$. In the spirit of high-level mathematics, we define this object via its relationship to all other sheaves, rather than by its internal construction.

Let $\iota : \mathsf{Sh}(X) \hookrightarrow \mathsf{PSh}(X)$ be the inclusion (forgetful) functor.

Definition 20 Sheafification.

The **sheafification** of a presheaf $\mathscr{F}$ is a sheaf $\mathscr{F}^+$ together with a morphism of presheaves $\theta : \mathscr{F} \to \mathscr{F}^+$ satisfying the following **universal property**: For any sheaf $\mathscr{G}$ and any presheaf morphism $\phi : \mathscr{F} \to \mathscr{G}$, there exists a unique morphism of sheaves $\psi : \mathscr{F}^+ \to \mathscr{G}$

such that the diagram commutes:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\theta"] \arrow[dr, "\phi"'] & \mathscr{F}^+ \arrow[d, dashed, "\psi"] \\
& \mathscr{G}
\end{tikzcd}$$

Note 21 Adjunction.

In categorical terms, sheafification is the **left adjoint** to the inclusion functor ι:

$$\mathrm{Hom}_{\mathsf{Sh}(X)}(\mathscr{F}^+, \mathscr{G}) \cong \mathrm{Hom}_{\mathsf{PSh}(X)}(\mathscr{F}, \iota(\mathscr{G})).$$

This definition guarantees that if $\mathscr{F}^+$ exists, it is unique up to unique isomorphism.

Motivated by the local-to-global principle, we construct the sheafification using the limit of compatible sections over open covers. This approach avoids the explicit construction of the bundle of stalks (the "espace étalé") and relies solely on the categorical properties of limits and colimits.

Definition 22 The Plus Construction.

Let X be a topological space and $\mathscr{F}$ a presheaf on X. For any open set $U \subseteq X$, let $\mathfrak{C}(U)$ be the set of all open covers $\mathcal{U} = \{U_i\}_{i\in I}$ of U. This set is directed by refinement: $\mathcal{V} \geq \mathcal{U}$ if every open set in $\mathcal{V}$ is contained in some open set in $\mathcal{U}$.

For a cover $\mathcal{U} = \{U_i\}$, we define the 0-th Čech cohomology group $\check{H}^0(\mathcal{U}, \mathscr{F})$ as the kernel of the difference map:

$$\check{H}^0(\mathcal{U}, \mathscr{F}) := \mathrm{Ker}\left(\prod_{i\in I} \mathscr{F}(U_i) \xrightarrow{d^0 - d^1} \prod_{i,j\in I} \mathscr{F}(U_i \cap U_j)\right),$$

where the maps are induced by restrictions: $d^0((s_i))|_{U_{ij}} = s_i|_{U_{ij}}$ and $d^1((s_i))|_{U_{ij}} = s_j|_{U_{ij}}$.

We define the presheaf $\mathscr{F}^+$ as the directed limit over the refinements:

$$\mathscr{F}^+(U) := \varinjlim_{\mathcal{U}\in\mathfrak{C}(U)} \check{H}^0(\mathcal{U}, \mathscr{F}).$$

The restriction maps for $\mathscr{F}^+$ are naturally induced by the restriction maps of $\mathscr{F}$.

There is a canonical presheaf morphism $\theta : \mathscr{F} \to \mathscr{F}^+$. For any $s \in \mathscr{F}(U)$, its image is defined by the trivial cover $\{U\}$ (where compatibility is automatic).

We provide the complete proof that the double-plus construction $\mathscr{F}^\# := (\mathscr{F}^+)^+$ constitutes the sheafification.

Definition 23 Separated Presheaf.

A presheaf $\mathscr{F}$ is called **separated** (or a monopresheaf) if for any open set U and any open cover $\{U_i\}$ of U, the first map in the sheaf sequence is injective:

$$0 \longrightarrow \mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i)$$

Lemma 24 Plus Construction Yields a Separated Presheaf.

For any presheaf $\mathscr{F}$, the presheaf $\mathscr{F}^+$ is separated.

Proof. Let U be an open set and $\{U_i\}_{i\in I}$ be an open cover of U. Suppose $s \in \mathscr{F}^+(U)$ is a section such that $s|_{U_i} = 0$ for all i. We must show $s = 0$.

By the definition of the directed limit, s is represented by an element $\sigma \in \check{H}^0(\mathcal{V}, \mathscr{F})$ for some cover $\mathcal{V} = \{V_j\}_{j\in J}$ of U. The condition $s|_{U_i} = 0$ implies that for each i, the restriction of σ to the refinement on U_i becomes zero in the directed limit over U_i.

Since the limit is directed, this means there exists a refinement of the cover restricted to U_i where the representative vanishes explicitly. Since $\{U_i\}$ covers U, we can combine these local refinements into a single global refinement $\mathcal{W}$ of $\mathcal{V}$ (essentially consisting of open sets contained in $V_j \cap U_i$). On this refinement $\mathcal{W}$, the representative of s is zero. Thus s represents the zero element in $\varinjlim \check{H}^0(\cdot, \mathscr{F})$. Hence, $\mathscr{F}^+$ is separated. □

Lemma 25 Plus Construction on Separated Presheaf Yields Sheaf.

If $\mathscr{F}$ is a separated presheaf, then $\mathscr{F}^+$ is a sheaf.

Proof. Since $\mathscr{F}$ is separated, the canonical map $\mathscr{F} \to \mathscr{F}^+$ is injective. We know from the previous lemma that $\mathscr{F}^+$ is separated, so we only need to verify the gluing property.

Let $\{U_i\}$ be a cover of U, and let $s_i \in \mathscr{F}^+(U_i)$ be sections such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. Each s_i is represented by a family of sections in $\mathscr{F}$ defined on some cover $\mathcal{V}_i$ of U_i. Let $\mathcal{W} = \bigcup_i \mathcal{V}_i$. This is a cover of U.

Because $\mathscr{F}$ is separated, the compatibility of the elements s_i in $\mathscr{F}^+$ implies that the underlying representatives in $\mathscr{F}$ (defined on the fine grid $\mathcal{W}$) satisfy the cocycle condition on overlaps (potentially after passing to a further refinement). Specifically, since $\mathscr{F}$ is separated, "being compatible in the limit" descends to "being compatible on a sufficiently fine cover".

Therefore, the collection of all these local representatives defines an element in $\check{H}^0(\mathcal{W}, \mathscr{F})$. This element defines a global section $s \in \mathscr{F}^+(U)$. By construction, $s|_{U_i} = s_i$. Thus $\mathscr{F}^+$ is a sheaf. □

Theorem 26 Construction and Uniqueness.

Let $\mathscr{F}$ be a presheaf. Let $\mathscr{F}^\# := (\mathscr{F}^+)^+$. Then:

1. $\mathscr{F}^\#$ is a sheaf.
2. The natural map $\theta : \mathscr{F} \to \mathscr{F}^\#$ satisfies the universal property of sheafification.

Proof. **1. Sheaf Property:** By Lemma 1, $\mathscr{F}^+$ is a separated presheaf. By Lemma 2, applying the plus construction to a separated presheaf yields a sheaf. Thus $(\mathscr{F}^+)^+$ is a sheaf.

2. Universal Property: Let $\mathscr{G}$ be a sheaf and $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism. We wish to factor this uniquely through $\mathscr{F} \to \mathscr{F}^+$. Note that since $\mathscr{G}$ is a sheaf, $\mathscr{G} \cong \mathscr{G}^+$. (The limit of compatible sections of a sheaf is just the sections of the sheaf itself). The morphism ϕ induces a morphism on the limits:

$$\phi^+ : \mathscr{F}^+ \longrightarrow \mathscr{G}^+ \cong \mathscr{G}.$$

If we iterate this, we get $\phi^\# : \mathscr{F}^\# \to \mathscr{G}^\# \cong \mathscr{G}$. This gives existence.

For uniqueness, suppose we have two extensions $\psi_1, \psi_2 : \mathscr{F}^\# \to \mathscr{G}$. Their difference allows us to define a kernel. Since $\mathscr{F} \to \mathscr{F}^\#$ induces an isomorphism on stalks (as proven in the Proposition on stalks), and a morphism of sheaves is determined by its action on stalks, if ψ_1 and ψ_2 agree on $\mathscr{F}$, they agree on all stalks of $\mathscr{F}^\#$, and thus must be identical. □

Note 27 Why Double Plus?.

In many modern treatments (e.g., restricted to Noetherian schemes or specific topologies), one step often suffices, or the limit is taken differently. However, the double-plus construction is the rigorous categorical solution for arbitrary topological spaces.

We now introduce the standard algebraic operations for sheaves of abelian groups (or modules).

Note 28 Stalk-Wise Behavior.

The definitions below may seem technical, especially the distinction between presheaf operations and sheaf operations.

Do not get bogged down in the construction details on your first reading.

The most important takeaway is the **behavior on stalks**. thanks to our previous results, we know that for any operation defined below (kernel, image, quotient, etc.), the stalk of the result is simply the corresponding algebraic operation on the stalks of the inputs.

$$(\mathscr{F} \oplus \mathscr{G})_x \cong \mathscr{F}_x \oplus \mathscr{G}_x, \quad (\operatorname{coker} \phi)_x \cong \operatorname{coker}(\phi_x), \quad \text{etc.}$$

This "stalk-wise principle" is your primary tool for calculation.

Definition 29 Ringed Space.

A **ringed space** is a pair $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a sheaf of rings on X, called the **structure sheaf**.

Definition 30 Morphism of Ringed Spaces.

A morphism of ringed spaces $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ consists of:

1. A continuous map $f : X \to Y$.

2. A homomorphism of sheaves of rings on X:

$$f^\# : f^{-1}\mathscr{O}_Y \longrightarrow \mathscr{O}_X.$$

(Note: By the adjunction $f^{-1} \dashv f_*$, this is equivalent to giving a map $\mathscr{O}_Y \to f_*\mathscr{O}_X$. We prefer the f^{-1} formulation as it represents "pulling back functions".)

In geometry, we care about functions' values at points. For a ringed space, the "value" of a section $s \in \mathscr{O}_X(U)$ at $x \in U$ is its image in the residue field.

Definition 31 Locally Ringed Space.

A ringed space $(X, \mathscr{O}_X)$ is a **locally ringed space** if for every point $x \in X$, the stalk $\mathscr{O}_{X,x}$ is a **local ring**. Let $\mathfrak{m}_x$ denote the unique maximal ideal of $\mathscr{O}_{X,x}$. The field $\kappa(x) := \mathscr{O}_{X,x}/\mathfrak{m}_x$ is called the **residue field** at x.

This condition captures the idea that "functions not vanishing at x are invertible near x".

Definition 32 Morphism of Locally Ringed Spaces.

A morphism of locally ringed spaces is a morphism of ringed spaces $(f, f^\#)$ such that for every $x \in X$, the induced map on stalks

$$f_x^\# : \mathscr{O}_{Y,f(x)} \longrightarrow \mathscr{O}_{X,x}$$

is a **local homomorphism** of local rings. This means $f_x^\#(\mathfrak{m}_{f(x)}) \subseteq \mathfrak{m}_x$.

Note 33 Why Local Homomorphisms?.

eometrically, $s \in \mathfrak{m}_{f(x)}$ means "the function s vanishes at $f(x)$". The condition $f^\#(s) \in \mathfrak{m}_x$ means "the pulled-back function vanishes at x". Without this condition, we could map the zero function to a non-zero constant, which violates geometric intuition.

2.3 More Presheaves and Sheaves

In these cases, the "naive" operation on presheaves (defined section-wise) automatically yields a sheaf.

Definition 34 Subsheaf and Kernel.

Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.

i. The **kernel** $\ker(\phi)$ is the sheaf defined by $U \mapsto \ker(\phi_U : \mathscr{F}(U) \to \mathscr{G}(U))$.

ii. A sheaf $\mathscr{F}$ is a **subsheaf** of $\mathscr{G}$ if for every U, $\mathscr{F}(U)$ is a subgroup of $\mathscr{G}(U)$ compatible with restriction.

It is a standard verification that these are indeed sheaves.

Definition 35 Product and Inverse Limit.

Let $\{\mathscr{F}_i\}_{i\in I}$ be a family of sheaves. The **product sheaf** $\prod \mathscr{F}_i$ is defined by $U \mapsto \prod(\mathscr{F}_i(U))$. More generally, the **inverse limit** (projective limit) $\varprojlim \mathscr{F}_i$ of a diagram of sheaves is defined section-wise: $(\varprojlim \mathscr{F}_i)(U) = \varprojlim(\mathscr{F}_i(U))$. These operations preserve the sheaf property.

Here, the naive presheaf operation often fails to produce a sheaf (it fails the gluing axiom). The correct sheaf operation is obtained by applying the **sheafification functor** $(-)^+$ to the presheaf result.

Definition 36 Cokernel and Quotient.

Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism. The **cokernel presheaf** is defined by $U \mapsto \operatorname{coker}(\phi_U) = \mathscr{G}(U)/\operatorname{im}(\phi_U)$. This is usually not a sheaf. The **cokernel sheaf** is its sheafification:

$$\operatorname{coker}(\phi) := (\text{presheaf cokernel})^+$$

If $\mathscr{F}$ is a subsheaf of $\mathscr{G}$, the **quotient sheaf** $\mathscr{G}/\mathscr{F}$ is the cokernel of the inclusion.

Definition 37 Image.

The **image sheaf** $\operatorname{im}(\phi)$ is the sheafification of the presheaf image $U \mapsto \operatorname{im}(\phi_U)$. Alternatively, it is the kernel of the map to the cokernel: $\operatorname{im}(\phi) = \ker(\mathscr{G} \to \operatorname{coker}(\phi))$.

Definition 38 Direct Sum and Direct Limit.

Let $\{\mathscr{F}_i\}_{i\in I}$ be a family of sheaves. The **direct sum** (coproduct) $\bigoplus \mathscr{F}_i$ is the sheafification of the presheaf direct sum $U \mapsto \bigoplus(\mathscr{F}_i(U))$. (Note: If I is finite, the presheaf sum is already a sheaf and coincides with the product). More generally, the **direct limit** (inductive limit) $\varinjlim \mathscr{F}_i$ is the sheafification of the section-wise direct limit.

Example 39 Cokernel Sheaf vs. Presheaf.

Consider the map of sheaves of holomorphic functions on $\mathbb{C}^*$:

$$\exp : \mathscr{O} \longrightarrow \mathscr{O}^*, \quad f \longmapsto e^{2\pi i f}.$$

This map is surjective on stalks (locally, every non-zero function has a logarithm). Thus, as a map of sheaves, it is an epimorphism (its cokernel is 0). However, on the global open set $U = \mathbb{C}^*$, the function $z \in \mathscr{O}^*(U)$ is not in the image of the map on sections (no global logarithm). Thus, the presheaf cokernel is non-zero at U, but the sheaf cokernel is zero. Sheafification kills this "local obstruction".

The following proposition is the computational heart of sheaf theory. It asserts that the operation of taking stalks commutes with all finite algebraic operations. In categorical terms, the stalk functor is **exact**.

Proposition 40 Stalks of Derived Sheaves.

Let X be a topological space. Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups, and let $\{\mathscr{F}_i\}$ be a family of sheaves. For any point $x \in X$, the stalks of the derived sheaves are canonically isomorphic to the corresponding operations on the stalks of $\mathscr{F}$ and $\mathscr{G}$:

1. **Kernel:** $(\ker\phi)_x \cong \ker(\phi_x)$.
2. **Cokernel:** $(\operatorname{coker}\phi)_x \cong \operatorname{coker}(\phi_x)$.
3. **Image:** $(\operatorname{im}\phi)_x \cong \operatorname{im}(\phi_x)$.
4. **Quotient:** If $\mathscr{F}' \subseteq \mathscr{F}$ is a subsheaf, $(\mathscr{F}/\mathscr{F}')_x \cong \mathscr{F}_x/\mathscr{F}'_x$.
5. **Direct Sum:** $(\bigoplus \mathscr{F}_i)_x \cong \bigoplus(\mathscr{F}_i)_x$.
6. **Finite Product:** If the index set is finite, $(\prod \mathscr{F}_i)_x \cong \prod(\mathscr{F}_i)_x$.

Consequently, a sequence of sheaves $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ is exact if and only if the sequence of stalks $\mathscr{F}'_x \to \mathscr{F}_x \to \mathscr{F}''_x$ is exact for every $x \in X$.

Proof. The proof relies on two fundamental facts about direct limits (colimits):

(A) Direct limits commute with finite limits (and thus with kernels).

(B) Direct limits commute with colimits (and thus with cokernels and direct sums).

We also need a crucial lemma linking sheaves to presheaves.

Proposition 41 Stalks of Sheafification.

Let $\mathscr{F}$ be a presheaf and let $\mathscr{F}^+$ be the presheaf defined above. For any point $x \in X$, the canonical morphism $\theta : \mathscr{F} \to \mathscr{F}^+$ induces an isomorphism on the stalks:

$$\theta_x : \mathscr{F}_x \xrightarrow{\sim} (\mathscr{F}^+)_x.$$

Consequently, the sheafification morphism $\mathscr{F} \to \mathscr{F}^\#$ also induces an isomorphism on stalks.

Proof. Recall that the stalk is a directed limit over open neighborhoods: $\mathscr{F}_x = \varinjlim_{U \ni x} \mathscr{F}(U)$. Thus, we examine:

$$(\mathscr{F}^+)_x = \varinjlim_{U \ni x} \mathscr{F}^+(U) = \varinjlim_{U \ni x} \left(\varinjlim_{\mathcal{U} \in \mathfrak{C}(U)} \check{H}^0(\mathcal{U}, \mathscr{F}) \right).$$

Since colimits commute with colimits, we can combine the limits. An element in $(\mathscr{F}^+)_x$ is represented by a pair (U, σ), where $U \ni x$ and $\sigma \in \mathscr{F}^+(U)$. By definition, σ is represented by a family $(s_i)_{i \in I}$ on a cover $\mathcal{U} = \{U_i\}$ of U such that s_i and s_j agree on overlaps.

Since $x \in U = \bigcup U_i$, there exists some index k such that $x \in U_k$. The element $s_k \in \mathscr{F}(U_k)$ represents a germ in $\mathscr{F}_x$. Conversely, any germ $s_x \in \mathscr{F}_x$ comes from a section $s \in \mathscr{F}(V)$ for some $V \ni x$. We can view this as a section in $\mathscr{F}^+(V)$ using the trivial cover.

Crucially, two elements in $\mathscr{F}^+(U)$ define the same germ at x if and only if they agree on a smaller neighborhood $W \ni x$. By the definition of the directed limit in the construction of $\mathscr{F}^+$, equality implies they eventually become equal on a common refinement. In the stalk limit, we can always refine neighborhoods to be contained in a single element of any cover. Thus, the map is both injective and surjective. □

Now we prove the items in the Proposition.

1. Kernel (Limit type): The kernel sheaf is defined section-wise: $(\ker \phi)(U) = \ker(\phi_U)$. Thus $\ker \phi$ is the same as the presheaf kernel.

$$(\ker \phi)_x = \varinjlim_{U \ni x} \ker(\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)).$$

Since the direct limit of exact sequences of abelian groups is exact, we have:

$$\varinjlim \ker(\phi(U)) \cong \ker(\varinjlim \phi(U)) = \ker(\phi_x).$$

2. Cokernel (Colimit type): The cokernel sheaf is the sheafification of the presheaf cokernel $\mathcal{P} = \operatorname{coker}^{pre}(\phi)$. By the Lemma, the stalk of the sheafification is the stalk of the presheaf.

$$(\operatorname{coker} \phi)_x \cong (\operatorname{coker}^{pre} \phi)_x = \varinjlim_{U \ni x} \operatorname{coker}(\phi(U)).$$

Since direct limits commute with cokernels (both are colimits):

$$\varinjlim \operatorname{coker}(\phi(U)) \cong \operatorname{coker}(\varinjlim \phi(U)) = \operatorname{coker}(\phi_x).$$

3. Image: Since $\operatorname{im}(\phi) = \ker(\mathscr{G} \to \operatorname{coker}(\phi))$, and taking stalks commutes with both kernels and cokernels (proven above), it commutes with images.

4. Quotient: The quotient $\mathscr{F}/\mathscr{F}'$ is simply $\operatorname{coker}(\mathscr{F}' \hookrightarrow \mathscr{F})$. The result follows from (2).

5. Direct Sum: The direct sum is the sheafification of the presheaf direct sum.

$$\left(\bigoplus \mathscr{F}_i\right)_x \cong \left(\overset{pre}{\bigoplus} \mathscr{F}_i\right)_x = \varinjlim_{U} \bigoplus_{i} \mathscr{F}_i(U) \cong \bigoplus_{i} \varinjlim_{U} \mathscr{F}_i(U) = \bigoplus_{i} (\mathscr{F}_i)_x.$$

(Direct limits commute with direct sums).

6. Finite Product: For finite index sets, the product sheaf coincides with the direct sum sheaf (biproduct). Thus it follows from (5).

Warning: For infinite products, $(\prod \mathscr{F}_i)_x$ is generally **not** isomorphic to $\prod (\mathscr{F}_i)_x$, because direct limits do not commute with infinite products. □

2.4 Properties of Sheaves and Morphisms

Theorem 42 Properties of Morphisms of Sheaves.

Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism. Let $\mathscr{F} \xrightarrow{\phi} \mathscr{G} \xrightarrow{\psi} \mathscr{H}$ be a sequence. The following table summarizes the necessary and sufficient conditions for various properties.

Property	Category of Presheaves	Category of Sheaves
Monomorphism (Injectivity)	$\forall U, \phi(U)$ is injective	$\forall U, \phi(U)$ is injective $\iff \forall x, \phi_x$ is injective
Epimorphism (Surjectivity)	$\forall U, \phi(U)$ is surjective	$\forall x, \phi_x$ is surjective $\iff$ Locally surjective on sections
Isomorphism (Bijectivity)	$\forall U, \phi(U)$ is bijective	$\forall x, \phi_x$ is bijective $\iff \forall U, \phi(U)$ is bijective
Exactness at $\mathscr{G}$	$\forall U, \operatorname{Im} \phi(U) = \ker \psi(U)$	$\forall x, \operatorname{Im} \phi_x = \ker \psi_x$

We now prove the non-trivial implications and provide counter-examples for the false ones.

1. Injectivity (Monomorphisms)

i. **Presheaves:** By definition of the category of functors, ϕ is a monomorphism iff $\phi(U)$ is injective for all U.

ii. **Sheaves:** Since the kernel sheaf is defined section-wise ($\ker(\phi)(U) = \ker(\phi(U))$), a morphism of sheaves is injective iff it is injective on all sections. ϕ is injective iff $\ker(\phi) = 0$. Since $(\ker \phi)_x = \ker(\phi_x)$, this is equivalent to $\ker(\phi_x) = 0$ for all x. For sheaves, Section-Injectivity $\iff$ Stalk-Injectivity.

2. Surjectivity (Epimorphisms)

i. **Presheaves:** ϕ is an epimorphism iff $\phi(U)$ is surjective for all U.

ii. **Sheaves:** This is the subtle case. The cokernel sheaf is the *sheafification* of the presheaf cokernel. ϕ is surjective iff $\operatorname{coker}(\phi) = 0$. Since $(\operatorname{coker} \phi)_x = \operatorname{coker}(\phi_x)$, surjectivity is equivalent to ϕ_x being surjective for all x.

Warning: Stalk-Surjectivity does NOT imply Section-Surjectivity. See the Exponential Sequence example ($e^{2\pi iz} : \mathscr{O} \to \mathscr{O}^*$). It is surjective on stalks (locally solvable), but not on global sections on $\mathbb{C}^*$.

Surjectivity of sheaves means "locally surjective". For any $t \in \mathscr{G}(U)$, there is a cover $U = \bigcup U_i$ and sections $s_i \in \mathscr{F}(U_i)$ such that $\phi(s_i) = t|_{U_i}$.

3. Exactness

i. **Presheaves:** Exactness is defined pointwise on sections.

ii. **Sheaves:** Exactness is defined by $\operatorname{im}(\phi) = \ker(\psi)$ in the category of sheaves. Since stalk functor is exact, this is equivalent to exactness on stalks.

If a sequence of sheaves is exact as presheaves (section-wise exact), it is exact as sheaves (stalk-wise exact). The converse is false (due to the failure of surjectivity on sections).

Corollary 43 Exactness Criterion for Sheaves.

A sequence of sheaves $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is exact if and only if:

1. For every open U, $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U)$ is exact (Injectivity and Kernel condition holds section-wise).
2. The map $\mathscr{F} \to \mathscr{F}''$ is locally surjective (Stalk-surjectivity).

We now study how sheaves move between different topological spaces via a continuous map $f : X \to Y$. The relationship is best understood through the lens of adjoint functors.

The direct image is the most straightforward operation: we "push" sections forward.

Definition 44 Direct Image.

Let $f : X \to Y$ be a continuous map and $\mathscr{F}$ a sheaf on X. The **direct image sheaf** $f_*\mathscr{F}$ on Y is defined by:

$$(f_*\mathscr{F})(V) := \mathscr{F}(f^{-1}(V)) \quad \text{for all open } V \subseteq Y.$$

Restriction maps are induced naturally by the restrictions of $\mathscr{F}$.

Proposition 45 Left Exactness.

The functor $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$ is left exact. That is, it preserves kernels and exact sequences of the form $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$.

Proof. This follows immediately because limits (like kernels) in the category of sheaves are computed section-wise, and the definition of f_* is just re-indexing sections. □

The inverse image is subtler. We want a functor $f^{-1} : \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$ that is **left adjoint** to f_*.

$$\operatorname{Hom}_{\mathsf{Sh}(X)}(f^{-1}\mathscr{G}, \mathscr{F}) \cong \operatorname{Hom}_{\mathsf{Sh}(Y)}(\mathscr{G}, f_*\mathscr{F}).$$

This adjunction dictates how f^{-1} must be constructed.

Definition 46 Inverse Image.

Let $\mathscr{G}$ be a sheaf on Y. We define $f^{-1}\mathscr{G}$ in two steps:

1. Define the **presheaf** $f^{+}\mathscr{G}$ on X by the direct limit over neighborhoods:
$$(f^{+}\mathscr{G})(U) := \varinjlim_{V \supseteq f(U)} \mathscr{G}(V).$$
2. Define $f^{-1}\mathscr{G}$ as the **sheafification** of this presheaf:
$$f^{-1}\mathscr{G} := (f^{+}\mathscr{G})^{+}.$$

Theorem 47 Adjunction and Stalks.

The functor f^{-1} is the left adjoint to f_*. As a consequence (by LAPC), f^{-1} is **right exact** and preserves stalks:
$$(f^{-1}\mathscr{G})_x \cong \mathscr{G}_{f(x)}.$$

We aim to establish a natural bijection:
$$\mathrm{Hom}_{\mathsf{Sh}(X)}(f^{-1}\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathsf{Sh}(Y)}(\mathscr{G}, f_*\mathscr{F}).$$

Recall that $f^{-1}\mathscr{G}$ is the sheafification of the presheaf $f^{+}\mathscr{G}$. By the universal property of sheafification (Adjunction of $(-)^{+}$ and inclusion), a morphism from a sheafification to a sheaf is determined uniquely by a morphism from the underlying presheaf:
$$\mathrm{Hom}_{\mathsf{Sh}(X)}((f^{+}\mathscr{G})^{+}, \mathscr{F}) \cong \mathrm{Hom}_{\mathsf{PSh}(X)}(f^{+}\mathscr{G}, \mathscr{F}).$$

Thus, the problem reduces to proving the adjunction at the **presheaf level**:
$$\mathrm{Hom}_{\mathsf{PSh}(X)}(f^{+}\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathsf{PSh}(Y)}(\mathscr{G}, f_*\mathscr{F}).$$

Proof. Let $\Phi : \mathrm{Hom}_{\mathsf{PSh}(X)}(f^{+}\mathscr{G}, \mathscr{F}) \to \mathrm{Hom}_{\mathsf{PSh}(Y)}(\mathscr{G}, f_*\mathscr{F})$ and Ψ be its inverse.

Step 1: Construction of Φ (Left to Right): Let $\alpha : f^{+}\mathscr{G} \to \mathscr{F}$ be a morphism of presheaves on X. We want to construct $\beta : \mathscr{G} \to f_*\mathscr{F}$. For any open $V \subseteq Y$, we need a map $\beta_V : \mathscr{G}(V) \to (f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$. Recall that $(f^{+}\mathscr{G})(f^{-1}(V)) = \varinjlim_{W \supseteq f(f^{-1}(V))} \mathscr{G}(W)$. Since $f(f^{-1}(V)) \subseteq V$, the open set V appears in the direct limit system. There is a canonical map into the colimit:
$$\rho_V : \mathscr{G}(V) \longrightarrow (f^{+}\mathscr{G})(f^{-1}(V)).$$
We define β_V as the composition with α evaluated at the open set $f^{-1}(V)$:
$$\beta_V := \alpha_{f^{-1}(V)} \circ \rho_V.$$

It is straightforward to check this is compatible with restrictions.

Step 2: Construction of Ψ (Right to Left): Let $\beta : \mathscr{G} \to f_*\mathscr{F}$ be a morphism of presheaves on Y. We want to construct $\alpha : f^{+}\mathscr{G} \to \mathscr{F}$. For any open $U \subseteq X$, we need a map $\alpha_U : (f^{+}\mathscr{G})(U) \to \mathscr{F}(U)$. By definition, $(f^{+}\mathscr{G})(U) = \varinjlim_{V \supseteq f(U)} \mathscr{G}(V)$. By the universal property of the direct limit, to define a map **from** the limit, we need to provide a family of compatible maps from each $\mathscr{G}(V)$ to $\mathscr{F}(U)$ for all $V \supseteq f(U)$. For such a V, we have the map $\beta_V : \mathscr{G}(V) \to \mathscr{F}(f^{-1}(V))$. Since $V \supseteq f(U)$, we have $f^{-1}(V) \supseteq f^{-1}(f(U)) \supseteq U$. Let res : $\mathscr{F}(f^{-1}(V)) \to \mathscr{F}(U)$ be the restriction map of the sheaf $\mathscr{F}$. We define the component map ϕ_V as the composite:

$$\mathscr{G}(V) \xrightarrow{\beta_V} \mathscr{F}(f^{-1}(V)) \xrightarrow{\text{res}} \mathscr{F}(U).$$

These maps are compatible with restrictions in V (due to the presheaf nature of β and $\mathscr{F}$). Thus, they induce a unique map α_U from the limit.

The constructions Φ and Ψ are clearly inverses, establishing the adjunction. □

We compute the stalk of $f^{-1}\mathscr{G}$ at a point $x \in X$. We use two facts:

1. The stalk of a sheafification is isomorphic to the stalk of the presheaf: $(f^{-1}\mathscr{G})_x \cong (f^{+}\mathscr{G})_x$.
2. Stalks are direct limits over neighborhoods.

Thus, we compute:

$$(f^{-1}\mathscr{G})_x = \varinjlim_{U \ni x} (f^{+}\mathscr{G})(U) = \varinjlim_{U \ni x} \left(\varinjlim_{V \supseteq f(U)} \mathscr{G}(V) \right).$$

This is a double direct limit. The index set is the set of pairs (U, V) such that $x \in U \subseteq X$ and $f(U) \subseteq V \subseteq Y$. Let $\mathscr{I}$ be this index set, ordered by reverse inclusion on both components. We want to show this limit is isomorphic to $\mathscr{G}_{f(x)} = \varinjlim_{W \ni f(x)} \mathscr{G}(W)$. Let $\mathcal{J} = \{W \subseteq Y \mid f(x) \in W\}$ be the index set for the latter.

Proof. We define a map of index sets $\pi : \mathscr{I} \to \mathcal{J}$ by $(U, V) \mapsto V$. Note that if $x \in U$ and $f(U) \subseteq V$, then $f(x) \in V$, so this is well-defined. To show the limits are isomorphic, it suffices to show that the image of π is **cofinal** in $\mathcal{J}$ and that the "diagonal" system matches the target system.

Is every neighborhood W of $f(x)$ "reached" by the system $\mathscr{I}$?

Let $W \in \mathcal{J}$ (so $f(x) \in W$). We need to find a pair $(U, V) \in \mathscr{I}$ such that $V \subseteq W$. Let $V = W$. Since f is continuous, $f^{-1}(W)$ is an open neighborhood of x. Let $U = f^{-1}(W)$.Then $x \in U$ and $f(U) = f(f^{-1}(W)) \subseteq W$.

Thus, the pair $(f^{-1}(W), W)$ is in $\mathscr{I}$, and its second component is W.This shows that as we shrink U around x and V around $f(U)$, the sets V run through a cofinal system of neighborhoods of $f(x)$.

$$\varinjlim_{U \ni x} \varinjlim_{V \supseteq f(U)} \mathscr{G}(V) \cong \varinjlim_{W \ni f(x)} \mathscr{G}(W) = \mathscr{G}_{f(x)}.$$

□

The adjunction (f^{-1}, f_*) comes with natural transformations.

1. **Unit (Pullback map):** $\eta : \mathscr{G} \to f_* f^{-1}\mathscr{G}$. For any open $V \subseteq Y$, this gives a map $\mathscr{G}(V) \to (f^{-1}\mathscr{G})(f^{-1}(V))$. **Meaning:** This is the natural map that takes a section s on Y and views it as a section on the preimage in X (the "pullback" of a function).

2. **Counit (Trace map):** $\varepsilon : f^{-1}f_*\mathscr{F} \to \mathscr{F}$. On stalks at x, this is the map $(f_*\mathscr{F})_{f(x)} \to \mathscr{F}_x$. **Meaning:** This corresponds to evaluating a section defined on a neighborhood of $f(x)$ (which lives in Y) at the specific point x (which lives in X).

The direct image and inverse image sheaf functors form an adjoint pair, from which we can easily deduce their exactness properties. However, it is worth noting, by referencing our previous discussion on the stalks of the inverse image sheaf, that it is, in fact, even exact.

Corollary 48 Exactness Summary.

We have:

i. f_* is Left Exact (preserves kernels). It is NOT right exact (hence cohomology).

ii. f^{-1} is Exact (preserves kernels and cokernels).

Proof. f^{-1} is right exact because it is a left adjoint. It is left exact because on stalks it is simply $(f^{-1}\mathscr{G})_x \cong \mathscr{G}_{f(x)}$, and taking stalks is exact. Thus f^{-1} is exact. □

2.5 Category of Modules over a Sheaf

Intuition 49 Coherent Sheaves.

The objective of this chapter is the rigorous investigation of coherent sheaves, which, together with their immediate correlates, constitute fundamental and operative conceptual tools in contemporary Algebraic Geometry.

Since a coherent sheaf is, by definition, a particular subclass of sheaves of modules, it is axiomatic to commence with a formal exposition of the theory of sheaves of modules over a ringed space (or a scheme).

This conceptual translation is in no way heuristically abstract; rather, it is a natural extension of the foundational work established previously. Indeed, having constructed a strict categorical and functorial correspondence between the notion of rings in Commutative Algebra and ringed spaces in Algebraic Geometry, the present chapter merely executes the "transport of structure" required to extend the concept of a module over a ring to that of a module over a sheaf of rings, thereby completing the Fundamental Analogy.

Definition 50 Sheaves of Modules.

Let $(X, \mathscr{O}_X)$ be a **ringed space**. A sheaf of $\mathscr{O}_X$-**modules**, denoted $\mathscr{F}$, is a sheaf of abelian groups on X satisfying the following structural axioms:

1. For every open subset $U \subset X$, the group $\mathscr{F}(U)$ possesses the structure of an $\mathscr{O}_X(U)$-module.
2. For every inclusion of open subsets $V \subset U$, the restriction homomorphism $\rho_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V)$ is a homomorphism of $\mathscr{O}_X(U)$-modules, and is **compatible** with the restriction of the ring structure.

That is, for all sections $s \in \mathscr{O}_X(U)$ and $t \in \mathscr{F}(U)$, the following identity holds for the restriction operation:

$$(s \cdot t)|_V = s|_V \cdot t|_V$$

Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$-modules. A **morphism** $\phi : \mathscr{F} \to \mathscr{G}$ in the category of $\mathscr{O}_X$-modules is a sheaf homomorphism such that, for every open set U, the map $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is a homomorphism of $\mathscr{O}_X(U)$-modules.

The collection of $\mathscr{O}_X$-modules, together with these morphisms, forms an **Abelian category**, denoted $\mathrm{Mod}(\mathscr{O}_X)$. The set of such morphisms is denoted $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$.

Definition 51 The Hom-Sheaf (Internal Hom).

The **Hom-sheaf** $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is the sheaf associated to the presheaf:

$$U \mapsto \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$$

Its stalk at $P \in X$ is given by the $\mathscr{O}_{X,P}$-module homomorphisms:

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_P \cong \mathrm{Hom}_{\mathscr{O}_{X,P}}(\mathscr{F}_P, \mathscr{G}_P)$$

Definition 52 The Tensor Product Sheaf.

The **tensor product** $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is the sheaf associated to the presheaf:

$$U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$$

The formation of the tensor product is local, respecting the module structure at the stalks:

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_P \cong \mathscr{F}_P \otimes_{\mathscr{O}_{X,P}} \mathscr{G}_P$$

Definition 53 Direct Sum and Direct Product.

Given a family of $\mathscr{O}_X$-modules $(\mathscr{F}_i)_{i \in I}$, the **direct sum** $\bigoplus_{i \in I} \mathscr{F}_i$ is the sheaf associated to the presheaf:

$$U \mapsto \bigoplus_{i \in I} \mathscr{F}_i(U)$$

The **direct product** $\prod_{i \in I} \mathscr{F}_i$ is the sheaf whose sections over U are the product of the individual sections:

$$U \mapsto \prod_{i \in I} \mathscr{F}_i(U)$$

In the category $\mathrm{Mod}(\mathscr{O}_X)$, the direct product presheaf is always a sheaf.

Definition 54 Limits in $\mathrm{Mod}(\mathscr{O}_X)$.

Let $(I, \leq)$ be a **directed set**.

i. The **direct limit** of a direct system $(\mathscr{F}_i, \phi_{ij})_{i \in I}$ is the sheaf associated to the presheaf:

$$U \mapsto \varinjlim_i \mathscr{F}_i(U)$$

ii. The **inverse limit** of an inverse system $(\mathscr{F}_i, \phi_{ji})_{i \in I}$ is the sheaf associated to the presheaf:

$$U \mapsto \varprojlim_i \mathscr{F}_i(U)$$

A note on Noetherian Spaces: When X is a **Noetherian topological space**, the presheaves corresponding to both the direct sum and the inverse limit are sheaves themselves. (This is established by noting that every open subset of a Noetherian space is quasi-compact.)

We illustrate the fundamental operations in the category $\mathrm{Mod}(\mathscr{O}_X)$ using two distinct ringed spaces: a smooth manifold and an affine scheme.

Example 55 Example I: Associated Sheaves on a Smooth Manifold.

Let X be a smooth manifold, and let $\mathscr{O}_X = C_X^\infty$ be the sheaf of smooth real-valued functions.

The **Tangent Sheaf** $\mathcal{T}_X$ is the sheaf of smooth vector fields on X.

i. Sections: For $U \subset X$, $\mathcal{T}_X(U)$ is the $C_X^\infty(U)$-module of vector fields on U.

ii. Compatibility Check: For $f \in C_X^\infty(U)$ and $V \in \mathcal{T}_X(U)$, the module axiom holds locally:

$$(f \cdot V)|_W = f|_W \cdot V|_W \quad \text{for } W \subset U.$$

Let Ω_X^1 be the sheaf of 1-forms (the cotangent sheaf). The Hom-sheaf $\mathscr{H}om_{\mathscr{O}_X}(\mathcal{T}_X, \Omega_X^1)$ has sections that correspond to $(0,2)$-tensor fields:

$$\mathscr{H}om_{C_X^\infty}(\mathcal{T}_X, \Omega_X^1) \cong \mathcal{T}_X^* \otimes_{C_X^\infty} \mathcal{T}_X^*$$

The sections of the tensor product $\mathcal{T}_X \otimes_{C_X^\infty} \mathcal{T}_X^*$ are the $(1,1)$-tensor fields. The stalk property is:

$$(\mathcal{T}_X \otimes_{C_X^\infty} \mathcal{T}_X^*)_P \cong \mathcal{T}_{X,P} \otimes_{C_{X,P}^\infty} \mathcal{T}_{X,P}^*$$

The direct sum of the tangent and cotangent sheaves corresponds to the Whitney sum of the vector bundles:

$$(\mathcal{T}_X \oplus \Omega_X^1)(U) \cong \mathcal{T}_X(U) \oplus \Omega_X^1(U)$$

Example 56 Example II: Associated Sheaves on an Affine Scheme.

Let R be a commutative ring and $X = \mathrm{Spec}(R)$ be the affine scheme. Let $\mathscr{O}_X$ be the structure sheaf $\mathscr{O}_R$.

For an R-module M, the **Associated Sheaf** $\widetilde{M}$ is defined by:

$$\widetilde{M}(D(f)) = M_f \quad \text{on distinguished open sets } D(f) \subset X.$$

The correspondence between module homomorphisms and sheaf morphisms holds:

$$\mathrm{Hom}_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N}) \cong \mathrm{Hom}_R(M, N)$$

The sheaf operations are induced by the module operations on the ring R:

$$\mathscr{H}om_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N}) \cong \widetilde{\mathrm{Hom}_R(M, N)}$$

$$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M \otimes_R N}$$

Let $R = k[[t]]$. Consider the inverse system of sheaves $\mathscr{F}_n = \mathcal{O}_X/\mathscr{I}_n$, where $\mathscr{I}_n = \widetilde{(t^n)}$. The global sections of the inverse limit sheaf are related to the completion of the ring:

$$\Gamma(X, \varprojlim_n \mathscr{F}_n) \cong \varprojlim_n \Gamma(X, \mathscr{F}_n) \cong \varprojlim_n R/(t^n) \cong R$$

We examine specific classes of $\mathcal{O}_X$-modules that serve as geometric analogues to key concepts in Commutative Algebra (e.g., Free Modules, Projective Modules, Finitely Generated Modules).

Definition 57 Free Modules of Sheaves.

An $\mathcal{O}_X$-module $\mathscr{F}$ is called **free** if it is isomorphic to a finite direct sum of copies of the structure sheaf:

$$\mathscr{F} \cong \bigoplus_{i=1}^{n} \mathcal{O}_X = \mathcal{O}_X^{\oplus n}.$$

The number of copies n is termed the **rank** of the free $\mathcal{O}_X$-module.

Definition 58 Locally Free.

An $\mathcal{O}_X$-module $\mathscr{F}$ is called **locally free** if there exists an open covering $\{U_i\}_{i \in I}$ of X such that for each i, the restriction $\mathscr{F}|_{U_i}$ is isomorphic to a free $\mathcal{O}_{U_i}$-module:

$$\mathscr{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n_i}.$$

The **rank** of $\mathscr{F}$ at a point $P \in X$ is defined by the rank of the stalk module, $\mathrm{rank}_{\mathcal{O}_{X,P}}(\mathscr{F}_P)$. For a locally free module, this rank is a locally constant function on X.

Definition 59 Invertible Modules and the Picard Group.

A locally free $\mathcal{O}_X$-module $\mathcal{L}$ is called **invertible** if it has rank one.

We establish that the isomorphism classes of invertible $\mathcal{O}_X$-modules form an abelian group under the tensor product, which is the definition of the **Picard group**, $\mathrm{Pic}(X)$. The proofs rely fundamentally on the local nature of sheaves and the known properties of R-modules (the **Algebraic Analogy**).

Let $\mathcal{L}$ and $\mathcal{M}$ be invertible $\mathcal{O}_X$-modules.

An $\mathcal{O}_X$-module $\mathscr{F}$ is invertible if and only if it is locally free of rank one. We first verify the locally free property. Since $\mathcal{L}$ and $\mathcal{M}$ are invertible, there exists an open covering $\{U_i\}$ such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$ and $\mathcal{M}|_{U_i} \cong \mathcal{O}_{U_i}$. The restriction of the tensor product is:

$$\mathscr{F}|_{U_i} \cong \mathcal{L}|_{U_i} \otimes_{\mathcal{O}_{U_i}} \mathcal{M}|_{U_i} \cong \mathcal{O}_{U_i} \otimes_{\mathcal{O}_{U_i}} \mathcal{O}_{U_i}$$

By the canonical result in Commutative Algebra, $R \otimes_R R \cong R$, we deduce $\mathscr{F}|_{U_i} \cong \mathcal{O}_{U_i}$. Thus, $\mathscr{F}$ is locally free.

Next, we establish the rank one property by examining the stalk at any point $P \in X$. Using the locality of the tensor product, we have: $\mathscr{F}_P \cong \mathcal{L}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{M}_P$. Since $\mathcal{L}$ and $\mathcal{M}$ are invertible, their stalks $\mathcal{L}_P$ and $\mathcal{M}_P$ are free $\mathcal{O}_{X,P}$-modules of rank one. Setting $R = \mathcal{O}_{X,P}$, we find $\mathscr{F}_P \cong R \otimes_R R \cong R$. Therefore, $\mathrm{rank}_P(\mathscr{F}) = 1$ for all $P \in X$.

Since $\mathscr{F}$ is locally free of rank one, the claim is established.

For an invertible $\mathscr{O}_X$-module $\mathcal{L}$, the dual sheaf $\mathcal{L}^* = \mathscr{H}om_{\mathscr{O}_X}(\mathcal{L}, \mathscr{O}_X)$ is also invertible, and serves as the inverse element, satisfying the canonical isomorphism:

$$\mathcal{L} \otimes_{\mathscr{O}_X} \mathcal{L}^* \cong \mathscr{O}_X.$$

We first prove the invertibility of $\mathcal{L}^*$. Since $\mathcal{L}$ is invertible, there exists a covering $\{U_i\}$ such that $\mathcal{L}|_{U_i} \cong \mathscr{O}_{U_i}$. Applying the hom-sheaf property locally:

$$\mathcal{L}^*|_{U_i} \cong \mathscr{H}om_{\mathscr{O}_{U_i}}(\mathcal{L}|_{U_i}, \mathscr{O}_{U_i}) \cong \mathscr{H}om_{\mathscr{O}_{U_i}}(\mathscr{O}_{U_i}, \mathscr{O}_{U_i})$$

The algebraic result $\mathrm{Hom}_R(R, R) \cong R$ implies $\mathcal{L}^*|_{U_i} \cong \mathscr{O}_{U_i}$. Hence, $\mathcal{L}^*$ is locally free of rank one, proving its invertibility.

We next demonstrate the inverse property. We construct the canonical global isomorphism $\phi : \mathcal{L} \otimes_{\mathscr{O}_X} \mathcal{L}^* \to \mathscr{O}_X$ via the evaluation map. On any open set U, the map is given by: $\phi_U(s \otimes f) = f(s)$, where $s \in \mathcal{L}(U)$ and $f \in \mathcal{L}^*(U)$. This is an $\mathscr{O}_X(U)$-linear map. To confirm it is an isomorphism, we verify the induced map on the stalks P. The local map is $\mathcal{L}_P \otimes_{\mathscr{O}_{X,P}} \mathcal{L}^*_P \to \mathscr{O}_{X,P}$. Since $\mathcal{L}_P \cong \mathscr{O}_{X,P}$ (as R-modules), this reduces to the canonical isomorphism $R \otimes_R \mathrm{Hom}_R(R, R) \to R$, which is an isomorphism. By the local property of sheaves, the global map ϕ is a canonical isomorphism.

Therefore, $\mathcal{L}^*$ is the inverse element of $\mathcal{L}$ in $\mathrm{Pic}(X)$.

Definition 60 Modules of Finite Type.

An $\mathscr{O}_X$-module $\mathscr{F}$ is called of **finite type** if there exists an open covering $\{U_i\}_{i\in I}$ of X such that on each U_i, there is a surjection from a free $\mathscr{O}_{U_i}$-module of finite rank:

$$\mathscr{O}_{U_i}^{\oplus n_i} \to \mathscr{F}|_{U_i} \to 0, \quad \text{for some natural number } n_i.$$

Definition 61 Modules of Finite Presentation.

An $\mathscr{O}_X$-module $\mathscr{F}$ is called of **finite presentation** if there exists an open covering $\{U_i\}_{i\in I}$ of X such that on each U_i, there is an exact sequence of the form:

$$\mathscr{O}_{U_i}^{\oplus m} \to \mathscr{O}_{U_i}^{\oplus n} \to \mathscr{F}|_{U_i} \to 0,$$

where $\mathscr{O}_{U_i}^{\oplus m}$ and $\mathscr{O}_{U_i}^{\oplus n}$ are free $\mathscr{O}_{U_i}$-modules of finite ranks m and n, respectively.

Although finite presentation is algebraically a stronger condition than finite type, its precise advantages may not be immediately clear. Its most critical contribution, however, is guaranteeing the commutativity of the sheaf hom functor ($\mathscr{H}om$) with the stalk functor ($(-)_P$), establishing the fundamental isomorphism:

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_P \cong \mathrm{Hom}_{\mathscr{O}_{X,P}}(\mathscr{F}_P, \mathscr{G}_P)$$

Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules. We assume $\mathscr{F}$ is of finite presentation. By definition, $\mathscr{F}$ being finitely presented means there exists an exact sequence of $\mathscr{O}_X$-modules:

$$\mathscr{O}_X^m \xrightarrow{f} \mathscr{O}_X^n \xrightarrow{g} \mathscr{F} \to 0$$

Applying the left-exact functor $\mathscr{H}om_{\mathscr{O}_X}(-, \mathscr{G})$ yields the exact sequence:

$$0 \to \mathscr{H}om(\mathscr{F}, \mathscr{G}) \to \mathscr{H}om(\mathscr{O}_X^n, \mathscr{G}) \to \mathscr{H}om(\mathscr{O}_X^m, \mathscr{G})$$

For any $P \in X$, the functor of taking the stalk at P is exact. Applying $(-)_P$ yields the exact sequence of $\mathscr{O}_{X,P}$-modules:

$$0 \to (\mathscr{H}om(\mathscr{F}, \mathscr{G}))_P \xrightarrow{\alpha} (\mathrm{Hom}(\mathscr{O}_X^n, \mathscr{G}))_P \xrightarrow{\beta} (\mathrm{Hom}(\mathscr{O}_X^m, \mathscr{G}))_P$$

We utilize the canonical isomorphisms:

1. The stalk of a direct sum is the direct sum of the stalks: $(\mathscr{G}^s)_P \cong (\mathscr{G}_P)^s$.
2. The internal Hom of a free module is the power of the target module: $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X^s, \mathscr{G}) \cong \mathscr{G}^s$.
3. The Hom of a free module over the local ring is the power of the target module: $\mathrm{Hom}_{\mathscr{O}_{X,P}}(\mathscr{O}_{X,P}^s, \mathscr{G}_P) \cong (\mathscr{G}_P)^s$.

Combining these, we have a natural isomorphism γ_s for $s = n, m$:

$$\gamma_s : (\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X^s, \mathscr{G}))_P \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{O}_{X,P}}(\mathscr{O}_{X,P}^s, \mathscr{G}_P)$$

The exact sequence above induces an exact sequence of $\mathscr{O}_{X,P}$-modules on the stalks:

$$\mathscr{O}_{X,P}^m \xrightarrow{f_P} \mathscr{O}_{X,P}^n \xrightarrow{g_P} \mathscr{F}_P \to 0$$

Applying the left-exact functor $\mathrm{Hom}_{\mathscr{O}_{X,P}}(-, \mathscr{G}_P)$ yields the exact sequence:

$$0 \to \mathrm{Hom}(\mathscr{F}_P, \mathscr{G}_P) \xrightarrow{\alpha'} \mathrm{Hom}(\mathscr{O}_{X,P}^n, \mathscr{G}_P) \xrightarrow{\beta'} \mathrm{Hom}(\mathscr{O}_{X,P}^m, \mathscr{G}_P)$$

We construct the commutative diagram using sequences above:

$$\begin{tikzcd}
0 \arrow[r] & (\mathscr{H}om(\mathscr{F}, \mathscr{G}))_P \arrow[r, "\alpha"] \arrow[d, "\delta"] & (\mathscr{H}om(\mathscr{O}_X^n, \mathscr{G}))_P \arrow[r, "\beta"] \arrow[d, "\gamma_n", "\cong"'] & (\mathscr{H}om(\mathscr{O}_X^m, \mathscr{G}))_P \arrow[d, "\gamma_m", "\cong"'] \\
0 \arrow[r] & \mathrm{Hom}(\mathscr{F}_P, \mathscr{G}_P) \arrow[r, "\alpha'"] & \mathrm{Hom}(\mathscr{O}_{X,P}^n, \mathscr{G}_P) \arrow[r, "\beta'"] & \mathrm{Hom}(\mathscr{O}_{X,P}^m, \mathscr{G}_P)
\end{tikzcd}$$

Since γ_n and γ_m are isomorphisms and both rows are exact, the **Five Lemma** (or the 3×3 Lemma for left-exact sequences starting at 0) implies that the map $\delta : (\mathscr{H}om(\mathscr{F}, \mathscr{G}))_P \to \mathrm{Hom}(\mathscr{F}_P, \mathscr{G}_P)$ is an isomorphism.

Assume $\mathscr{F}$ and $\mathscr{G}$ are both of finite presentation, and let $\phi_P : \mathscr{F}_P \to \mathscr{G}_P$ be an isomorphism of $\mathscr{O}_{X,P}$-modules.

By the result of above, the element ϕ_P in $\mathrm{Hom}_{\mathscr{O}_{X,P}}(\mathscr{F}_P, \mathscr{G}_P)$ corresponds to a unique element $\phi_P^{\mathrm{germ}} \in (\mathscr{H}om(\mathscr{F}, \mathscr{G}))_P$. By the definition of the stalk, ϕ_P^{germ} is represented by a section $\phi \in \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})(U)$ for some open neighborhood U of P. This ϕ induces the map ϕ_P on the stalk at P.

Since ϕ_P is an isomorphism, its inverse $\phi_P^{-1} : \mathscr{G}_P \to \mathscr{F}_P$ exists in $\mathrm{Hom}_{\mathscr{O}_{X,P}}(\mathscr{G}_P, \mathscr{F}_P)$. Since $\mathscr{G}$ is also of finite presentation, we can apply Part (i) to $\mathrm{Hom}(\mathscr{G}, \mathscr{F})$. Thus, ϕ_P^{-1} corresponds to a unique element $\psi_P^{\mathrm{germ}} \in (\mathrm{Hom}(\mathscr{G}, \mathscr{F}))_P$. This ψ_P^{germ} is represented by a section $\psi \in \mathscr{H}om_{\mathscr{O}_X}(\mathscr{G}, \mathscr{F})(V)$ for some open neighborhood V of P.

Let $W = U \cap V$, which is a neighborhood of P. We have the stalk relations $\psi_P \circ \phi_P = \mathrm{id}_{\mathscr{F}_P}$ and $\phi_P \circ \psi_P = \mathrm{id}_{\mathscr{G}_P}$. By the definition of stalk equality, these relations imply that there exists a smaller neighborhood $U' \subseteq W$ of P such that on U':

$$\psi|_{U'} \circ \phi|_{U'} = \mathrm{id}_{\mathscr{F}|_{U'}} \quad \text{and} \quad \phi|_{U'} \circ \psi|_{U'} = \mathrm{id}_{\mathscr{G}|_{U'}}$$

Therefore, the restricted map $\phi|_{U'} : \mathscr{F}|_{U'} \to \mathscr{G}|_{U'}$ is an isomorphism of $\mathscr{O}_{U'}$-modules which induces the given stalk isomorphism ϕ_P.

Proposition 62 Finite Presentation and Hom-Sheaf Stalks.

Let $(X, \mathscr{O}_X)$ be a ringed space and let $\mathscr{F}$ and $\mathscr{G}$ be two $\mathscr{O}_X$-modules.

(i) If $\mathscr{F}$ is of finite presentation, then for every $P \in X$, we have

$$(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}))_P \cong \mathrm{Hom}_{\mathscr{O}_{X,P}}(\mathscr{F}_P, \mathscr{G}_P).$$

(ii) Suppose $\mathscr{F}$ and $\mathscr{G}$ are of finite presentation and $P \in X$. If there exists an isomorphism $\phi_P : \mathscr{F}_P \xrightarrow{\cong} \mathscr{G}_P$ of $\mathscr{O}_{X,P}$-modules, then there exists a neighborhood U of P and an isomorphism $\phi : \mathscr{F}|_U \xrightarrow{\cong} \mathscr{G}|_U$ of $\mathscr{O}_U$-modules inducing the isomorphism ϕ_P on stalks.

Intuition 63 Geometric Operations on Sheaves.

Before defining the geometric operations on sheaves, let us recall the corresponding operations on modules over rings. Let $\varphi : A \to B$ be a ring homomorphism. This corresponds geometrically to a morphism of affine schemes $f : \mathrm{Spec} B \to \mathrm{Spec} A$.

We have two natural ways to move modules between rings:

Algebraic Operation (Modules)	**Geometric Operation (Sheaves)**
Restriction of Scalars: Given a B-module N, we view it as an A-module via φ. $a \cdot n := \varphi(a)n$	**Direct Image** (f_*): Given an $\mathscr{O}_X$-module $\mathscr{F}$, we push it forward to Y. The $\mathscr{O}_X$-module structure on $\mathscr{F}$ induces an $\mathscr{O}_Y$-module structure on $f_*\mathscr{F}$ via $f^\#$.
Extension of Scalars: Given an A-module M, we construct a B-module via tensor product. $M \mapsto B \otimes_A M$	**Inverse Image** (f^*): Given an $\mathscr{O}_Y$-module $\mathscr{G}$, we pull it back to X. We pullback the topological sheaf and tensor it with the structure sheaf.

Let $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces.

Definition 64 Direct Image f_*.

Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. The **direct image** $f_*\mathscr{F}$ is the sheaf on Y defined by $(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$ for any open set $V \subseteq Y$. It becomes an $\mathscr{O}_Y$-module via the map $f^\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$. For $s \in \mathscr{O}_Y(V)$ and $m \in f_*\mathscr{F}(V)$, the action is defined by:

$$s \cdot m := f^\#(s) \cdot m \quad \text{(acting inside } \mathscr{F}(f^{-1}(V))\text{)}.$$

Definition 65 Inverse Image f^*.

Let $\mathscr{G}$ be an $\mathscr{O}_Y$-module. The **inverse image** $f^*\mathscr{G}$ is the $\mathscr{O}_X$-module defined by extensions of scalars from the topological inverse image $f^{-1}\mathscr{G}$. Explicitly:

$$f^*\mathscr{G} := f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X.$$

Here, $f^{-1}\mathscr{G}$ is naturally an $f^{-1}\mathscr{O}_Y$-module, and we tensor over the map of sheaves of rings $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ induced by adjointness from $f^\#$.

Proposition 66 Adjunction Formula.

The functor f^* is the left adjoint to the functor f_*. That is, for any $\mathscr{O}_Y$-module $\mathscr{G}$ and any $\mathscr{O}_X$-module $\mathscr{F}$, there is a natural bijection:

$$\mathscr{H}om_{\mathscr{O}_X}(f^*\mathscr{G}, \mathscr{F}) \cong \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{G}, f_*\mathscr{F}).$$

The proof follows from composing two fundamental adjunctions: the topological inverse image adjunction and the algebraic tensor-hom adjunction.

Step 1: Topological Adjunction.

First, we treat the modules over the pullback ring $f^{-1}\mathscr{O}_Y$. Recall that f^{-1} is the left adjoint to f_* at the level of sheaves of abelian groups (and explicitly compatible with module structures). Viewing $\mathscr{F}$ as an $f^{-1}\mathscr{O}_Y$-module via the map $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$, we have:

$$\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{G}, f_*\mathscr{F}) \cong \mathrm{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{G}, \mathscr{F}).$$

Step 2: Tensor-Hom Adjunction.

Next, we use the property of extension of scalars. For a ring homomorphism $A \to B$, an A-module M, and a B-module N, we have $\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_B(M \otimes_A B, N)$. Applying this sheaf-theoretically with $A = f^{-1}\mathscr{O}_Y$ and $B = \mathscr{O}_X$:

$$\mathrm{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{G}, \mathscr{F}) \cong \mathscr{H}om_{\mathscr{O}_X}(f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X, \mathscr{F}).$$

Combining these isomorphisms and using the definition $f^*\mathscr{G} := f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$, we obtain:

$$\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{G}, f_*\mathscr{F}) \cong \mathscr{H}om_{\mathscr{O}_X}(f^*\mathscr{G}, \mathscr{F}).$$

This completes the proof.

□

Note 67 Stalks of Inverse Image Modules.

The behavior of the inverse image on stalks reflects the tensor product of modules. For any point $p \in X$, let $q = f(p)$. We have:

$$(f^*\mathscr{G})_p \cong (f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X)_p \cong (f^{-1}\mathscr{G})_p \otimes_{(f^{-1}\mathscr{O}_Y)_p} \mathscr{O}_{X,p} \cong \mathscr{G}_q \otimes_{\mathscr{O}_{Y,q}} \mathscr{O}_{X,p}.$$

This aligns perfectly with the algebraic intuition of extension of scalars: $M \to M \otimes_A B$.

CHAPTER 3

Affine Schemes

3.1 The Zariski Topology on Affine Schemes

We now equip the set of prime ideals with a topology. The fundamental idea is to define "geometric shapes" (closed sets) as the loci where a collection of "functions" (elements of A) vanish.

Definition 1 Vanishing Locus.

Let A be a ring. For any subset $E \subseteq A$ (usually an ideal $\mathfrak{a}$), we define the *vanishing locus* $V(E)$ as the set of prime ideals containing E:

$$V(E) := \{\mathfrak{p} \in \operatorname{Spec} A \mid E \subseteq \mathfrak{p}\}.$$

If $\mathfrak{a}$ is the ideal generated by E, clearly $V(E) = V(\mathfrak{a})$.

Proposition 2 The Zariski Topology.

The collection of sets $\{V(\mathfrak{a}) \mid \mathfrak{a} \text{ is an ideal of } A\}$ satisfies the axioms of closed sets for a topology on $\operatorname{Spec} A$, called the *Zariski topology*.

1. $V(0) = \operatorname{Spec} A$ and $V(1) = \emptyset$.
2. *Order Reversing:* If $\mathfrak{a} \subseteq \mathfrak{b}$, then $V(\mathfrak{b}) \subseteq V(\mathfrak{a})$.
3. *Arbitrary Intersections:* For any family of ideals $\{\mathfrak{a}_i\}_{i \in I}$,

$$\bigcap_{i \in I} V(\mathfrak{a}_i) = V\left(\sum_{i \in I} \mathfrak{a}_i\right).$$

4. *Finite Unions:* For any two ideals $\mathfrak{a}, \mathfrak{b}$,

$$V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}).$$

Proof. (i) 0 is contained in every ideal, so $V(0)$ is the whole set. 1 is contained in no prime ideal (by definition of prime), so $V(1)$ is empty.

(ii) If $\mathfrak{p} \supseteq \mathfrak{b}$ and $\mathfrak{b} \supseteq \mathfrak{a}$, then $\mathfrak{p} \supseteq \mathfrak{a}$.

(iii) $\mathfrak{p} \in \bigcap V(\mathfrak{a}_i) \iff \forall i, \mathfrak{a}_i \subseteq \mathfrak{p} \iff \sum \mathfrak{a}_i \subseteq \mathfrak{p} \iff \mathfrak{p} \in V(\sum \mathfrak{a}_i)$.

(iv) Since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a}$ (and $\mathfrak{b}$), by (ii) we have $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$. Conversely, let $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$. Then $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime, this implies $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$. Thus $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$. □

The map $\mathfrak{a} \mapsto V(\mathfrak{a})$ is surjective onto closed sets but not injective. The following proposition clarifies that closed sets correspond one-to-one with *radical ideals*.

Proposition 3 Algebraic Nullstellensatz.

Let $\sqrt{\mathfrak{a}} = \{f \in A \mid f^n \in \mathfrak{a} \text{ for some } n\}$ denote the radical of $\mathfrak{a}$.

1. Spec$A = \emptyset$ if and only if $A = 0$.
2. $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.
3. *Intersection of Primes:* $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$.
4. The map $\mathfrak{a} \mapsto V(\mathfrak{a})$ induces a bijection between radical ideals of A and closed subsets of Spec$A$. Specifically, $V(\mathfrak{a}) \subseteq V(\mathfrak{b}) \iff \sqrt{\mathfrak{b}} \subseteq \sqrt{\mathfrak{a}}$.

Proof. (i) If $A = 0$, clearly Spec$A = \emptyset$. Conversely, if $A \neq 0$, let Σ be the set of all proper ideals of A. Since $1 \neq 0$, $(0) \in \Sigma$. Σ is partially ordered by inclusion. For any chain, the union is a proper ideal (it doesn't contain 1). By *Zorn's Lemma*, Σ has a maximal element $\mathfrak{m}$. Maximal ideals are prime, so $\mathfrak{m} \in \text{Spec}A \neq \emptyset$.

(ii) If $\mathfrak{a} \subseteq \mathfrak{p}$, then $f^n \in \mathfrak{a} \implies f^n \in \mathfrak{p} \implies f \in \mathfrak{p}$. So $V(\mathfrak{a}) \subseteq V(\sqrt{\mathfrak{a}})$. The reverse is obvious.

(iii) Clearly $\sqrt{\mathfrak{a}} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \supseteq \mathfrak{a}$, so $\sqrt{\mathfrak{a}} \subseteq \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$. Conversely, suppose $f \notin \sqrt{\mathfrak{a}}$. Consider the localization ring $B = (A/\mathfrak{a})_f$. In B, the image of f is a unit. Since f is not nilpotent in $A/\mathfrak{a}$, the ring B is not the zero ring ($0 \neq 1$ in B). By (i), Spec$B$ is non-empty. Let $\mathfrak{q} \in \text{Spec}B$. The contraction of $\mathfrak{q}$ to A via $A \to A/\mathfrak{a} \to B$ gives a prime ideal $\mathfrak{p}$ in A. By construction, $\mathfrak{a} \subseteq \mathfrak{p}$ (so $\mathfrak{p} \in V(\mathfrak{a})$) but $f \notin \mathfrak{p}$ (since f becomes a unit in B). Thus $f \notin \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$.

(iv) Follows immediately from (iii): $V(\mathfrak{a}) \subseteq V(\mathfrak{b}) \iff \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} \supseteq \bigcap_{\mathfrak{p} \in V(\mathfrak{b})} \mathfrak{p} \iff \sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$. □

The complement of $V((f))$ is denoted $D(f)$. These sets play the role of "coordinate charts".

Intuition 4 Compactness of the Zariski Topology.

An important characteristic of the Zariski topological space is that it is a compact space, meaning any open cover admits a finite subcover. However, the term "compact" was not adopted in the original algebraic geometry theory, as the French school at the time defined "compact" to be the finite covering property specifically in a Hausdorff space, using "quasi-compact" for the general topological case.

Proposition 5 Basis and Compactness.

For any $f \in A$, let $D(f) = \operatorname{Spec} A \setminus V((f)) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$.

1. The family $\{D(f)\}_{f \in A}$ forms a *basis* for the Zariski topology.
2. Each $D(f)$ is *quasi-compact*. In particular, $\operatorname{Spec} A = D(1)$ is quasi-compact.

Proof. (i) Let U be an open set. Then $U = \operatorname{Spec} A \setminus V(\mathfrak{a})$. We have $V(\mathfrak{a}) = V(\sum_{f \in \mathfrak{a}}(f)) = \bigcap_{f \in \mathfrak{a}} V((f))$. Taking complements, $U = \bigcup_{f \in \mathfrak{a}} D(f)$. Thus any open set is a union of principal open sets.

(ii) It suffices to prove this for $D(1) = \operatorname{Spec} A$, as $D(f)$ is homeomorphic to $\operatorname{Spec} A_f$. Let $\operatorname{Spec} A = \bigcup_{i \in I} D(f_i)$ be an open cover. Taking complements: $\emptyset = \bigcap_{i \in I} V((f_i)) = V(\sum_{i \in I}(f_i))$. Let $\mathfrak{a} = \sum(f_i)$ be the ideal generated by all f_i. $\emptyset = V(\mathfrak{a})$ implies $\mathfrak{a}$ is not contained in any prime ideal. By 3 (i), this means $\mathfrak{a} = A$, so $1 \in \mathfrak{a}$. Thus we can write 1 as a finite sum:

$$1 = a_1 f_{i_1} + \cdots + a_k f_{i_k}$$

for some indices $i_1, \ldots, i_k \in I$. This implies $V((1)) = V(f_{i_1}, \ldots, f_{i_k}) = \emptyset$, so $\operatorname{Spec} A = \bigcup_{j=1}^{k} D(f_{i_j})$. This finite subcover proves quasi-compactness. □

3.2 Topological Properties of Affine Schemes

Intuition 6 Topological Restrictions on Spectra of Rings.

In this section, we study several topological restrictions on the spectrum of a ring. Most of these are properties well-known to us from classical point-set topology. However, in the context of algebraic geometry—particularly in the special case of affine schemes—we obtain many interesting conclusions.

We can observe a very rigorous one-to-one correspondence between the topological properties of the geometry and the underlying algebraic properties. This does not imply that the affine schemes we study are trivial; rather, it provides us with more ways to understand the nature of ring theory and ideal theory.

At the same time, this will lay a solid foundation for our subsequent study of general schemes, as there is almost no difference in topological properties between a general scheme and an affine scheme. Furthermore, the properties of the schemes themselves are closely linked to the properties of their morphisms.

Proposition 7 Closure and Specialization.

Let $X = \operatorname{Spec} A$. For any point $x \in X$, let $\mathfrak{p}_x$ denote the corresponding prime ideal of A. The closure of the singleton set $\{x\}$ in the Zariski topology is given by

$$\overline{\{x\}} = V(\mathfrak{p}_x) = \{y \in X \mid \mathfrak{p}_y \supseteq \mathfrak{p}_x\}.$$

Consequently, x is a closed point if and only if $\mathfrak{p}_x$ is a maximal ideal of A. The relation defined by $x \le y \iff y \in \overline{\{x\}}$ constitutes a partial order on X, referred to as the *specialization* order.

Proof. By the definition of the Zariski topology, the closed sets of X are exactly the sets of the form $V(\mathfrak{a})$ for some ideal $\mathfrak{a} \subseteq A$. The closure $\overline{\{x\}}$ is the intersection of all closed sets containing x. A closed set $V(\mathfrak{a})$ contains x if and only if $\mathfrak{a} \subseteq \mathfrak{p}_x$. Since the correspondence between ideals and closed sets reverses inclusion, the smallest closed set containing x must correspond to the largest ideal contained in $\mathfrak{p}_x$ that defines a closed set.

Consider the closed set $V(\mathfrak{p}_x)$. Clearly, $x \in V(\mathfrak{p}_x)$ since $\mathfrak{p}_x \subseteq \mathfrak{p}_x$. Furthermore, for any other closed set $V(\mathfrak{a})$ containing x, we have established that $\mathfrak{a} \subseteq \mathfrak{p}_x$, which implies $V(\mathfrak{p}_x) \subseteq V(\mathfrak{a})$. Thus, $V(\mathfrak{p}_x)$ is contained in every closed set containing x, proving that $\overline{\{x\}} = V(\mathfrak{p}_x)$. The set-theoretic description $V(\mathfrak{p}_x) = \{y \in X \mid \mathfrak{p}_y \supseteq \mathfrak{p}_x\}$ follows directly from the definition of the vanishing locus.

For the second assertion, observe that a point x is closed if and only if $\overline{\{x\}} = \{x\}$. Based on the characterization above, this equality holds if and only if the set of prime ideals containing $\mathfrak{p}_x$ consists solely of $\mathfrak{p}_x$ itself. This condition is equivalent to $\mathfrak{p}_x$ being maximal with respect to set inclusion among all prime ideals. Since any proper ideal is contained in a maximal ideal, this implies $\mathfrak{p}_x$ must be a maximal ideal of A, i.e., $\mathfrak{p}_x \in \operatorname{MaxSpec} A$. Finally, the relation defined by inclusion of prime ideals is reflexive, antisymmetric, and transitive, thereby defining a partial order on the underlying topological space. □

Proposition 8 Canonical Homeomorphisms of Spectra.

Let A be a ring.

i. Let $f \in A$. The natural map $\varphi : A \longrightarrow A_f$ induces a homeomorphism between $\operatorname{Spec} A_f$ and the distinguished open set $D(f) \subseteq \operatorname{Spec} A$, where $D(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$.

ii. Let I be an ideal of A. The quotient map $\pi : A \longrightarrow A/I$ induces a homeomorphism between $\operatorname{Spec}(A/I)$ and the closed set $V(I) \subseteq \operatorname{Spec} A$, where $V(I) = \{\mathfrak{p} \in \operatorname{Spec} A \mid I \subseteq \mathfrak{p}\}$.

Proof. For the first assertion, recall that the prime ideals of the localization A_f are in one-to-one correspondence with the prime ideals of A that do not intersect the multiplicative set $S = \{1, f, f^2, \ldots\}$. This condition corresponds exactly to those $\mathfrak{p} \in \operatorname{Spec} A$ such that $f \notin \mathfrak{p}$, which is the set $D(f)$. Let $\psi = \operatorname{Spec}(\varphi) : \operatorname{Spec} A_f \longrightarrow \operatorname{Spec} A$ be the continuous map induced by φ. The image of ψ is exactly $D(f)$, and ψ is a bijection onto its image. To see that ψ is a homeomorphism onto $D(f)$, we must show it is an open map (or closed map) relative to the subspace topology. The open sets of $\operatorname{Spec} A_f$ are unions of distinguished open sets of the form $D(g/f^n)$ for $g \in A$. The bijection identifies $D(g/f^n) \subseteq \operatorname{Spec} A_f$ with $D(fg) \subseteq D(f) \subseteq \operatorname{Spec} A$. Since the basic open sets correspond, the topologies match.

For the second assertion, the prime ideals of the quotient ring A/I correspond bijectively to the prime ideals of A containing I. The induced map $\rho = \operatorname{Spec}(\pi) : \operatorname{Spec}(A/I) \longrightarrow \operatorname{Spec} A$ is continuous and its image is exactly $V(I)$. Furthermore, for any ideal $J \subseteq A$ containing I, the closed set $V(J/I)$ in $\operatorname{Spec}(A/I)$ is mapped to $V(J) \subseteq V(I)$. Since every closed set in $\operatorname{Spec}(A/I)$ is of this form, ρ is a closed map. Being a continuous bijection from $\operatorname{Spec}(A/I)$ to $V(I)$ that maps closed sets to closed sets, ρ is a homeomorphism. □

Definition 9 Irreducible Space.

A topological space X is said to be **irreducible** if X is non-empty and cannot be expressed as the union $X = Z_1 \cup Z_2$ of two proper closed subsets $Z_1, Z_2 \subsetneq X$. A subset of X is irreducible if it is irreducible under the subspace topology.

Proposition 10 Characterization of Irreducibility.

Let X be a non-empty topological space. The following conditions are equivalent:

(i) X is irreducible.

(ii) Every non-empty open subset of X is dense in X.

(iii) Any two non-empty open subsets of X have a non-empty intersection.

Proof. Assume (i) holds. Let U be a non-empty open subset. If U is not dense, then its closure $\overline{U}$ is a proper closed subset of X. Consequently, its complement U^c contains a non-empty open set, and importantly, $X = \overline{U} \cup (X \setminus U)$ would express X as a union of two proper closed sets (note that $X \setminus U$ is closed and proper since U is non-empty). This contradicts the irreducibility of X. Thus (i) implies (ii).

Next, assume (ii) holds. Let U_1, U_2 be two non-empty open sets. By assumption, U_1 is dense in X, which implies that U_1 intersects every non-empty open set. Therefore, $U_1 \cap U_2 \neq \emptyset$. Thus (ii) implies (iii).

Finally, assume (iii) holds. Suppose for the sake of contradiction that $X = Z_1 \cup Z_2$ where Z_1, Z_2 are proper closed subsets. Then their complements $U_1 = X \setminus Z_1$ and $U_2 = X \setminus Z_2$ are non-empty open sets. Since $Z_1 \cup Z_2 = X$, the intersection $U_1 \cap U_2 = (X \setminus Z_1) \cap (X \setminus Z_2) = X \setminus (Z_1 \cup Z_2)$ must be empty. This contradicts the assumption that any two non-empty open sets intersect. Thus (iii) implies (i). □

Proposition 11 Irreducibility in Spectra.

Let A be a ring and $X = \operatorname{Spec} A$.

i. A closed subset $Z \subseteq X$ is irreducible if and only if $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \in \operatorname{Spec} A$.

ii. There is a bijection between irreducible closed subsets of X and points of X. Specifically, every irreducible closed set Z has a unique **generic point** η such that $Z = \overline{\{\eta\}}$.

iii. The space $\operatorname{Spec} A$ is irreducible if and only if the nilradical $\sqrt{(0)}$ is a prime ideal. In particular, if A is an integral domain, $\operatorname{Spec} A$ is irreducible.

Proof. First, we verify that $V(\mathfrak{p})$ is irreducible. The closed set $V(\mathfrak{p})$ is homeomorphic to $\operatorname{Spec}(A/\mathfrak{p})$. Since $\mathfrak{p}$ is prime, $A/\mathfrak{p}$ is an integral domain. Thus, it suffices to prove statement (3) first. Suppose A is a domain; then (0) is prime. If $\operatorname{Spec} A = V(I) \cup V(J) = V(IJ)$, then every prime ideal contains IJ, so IJ is contained in the nilradical of A, which is

(0). Thus $IJ = 0$. Since A is a domain, this implies $I = 0$ or $J = 0$, so $V(I) = \operatorname{Spec} A$ or $V(J) = \operatorname{Spec} A$. Conversely, if $\operatorname{Spec} A$ is irreducible, consider the open sets $D(f)$ and $D(g)$. If their intersection $D(fg)$ is empty, then fg is nilpotent. If the nilradical $\sqrt{(0)}$ is prime, then $f \in \sqrt{(0)}$ or $g \in \sqrt{(0)}$, implying $D(f) = \emptyset$ or $D(g) = \emptyset$. This verifies the topological condition for irreducibility.

For (1) and (2), we observed that $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ is irreducible. Conversely, let Z be an irreducible closed set. We can write $Z = V(I)$ for some radical ideal I. If I were not prime, there would exist $f, g \notin I$ such that $fg \in I$. This would imply $Z \subseteq V(f) \cup V(g)$. Decomposing Z gives $Z = (Z \cap V(f)) \cup (Z \cap V(g))$. By irreducibility, Z must be contained in one, say $V(f)$, which implies f belongs to the radical of I, i.e., $f \in I$, a contradiction. Thus I must be prime, say $\mathfrak{p}$. The point corresponding to $\mathfrak{p}$ is the generic point η, and since the closure of a point $\mathfrak{p}$ is $V(\mathfrak{p})$, uniqueness follows from the fact that different primes define different closed sets (specifically, $\overline{\{\mathfrak{p}\}} = \overline{\{\mathfrak{q}\}} \implies \mathfrak{p} = \mathfrak{q}$). □

Proposition 12 Spectrum of Direct Products.

Let A and B be rings, and let $R = A \times B$ be their direct product. The spectrum of R is homeomorphic to the disjoint union of the spectra of the factors:

$$\operatorname{Spec}(A \times B) \cong \operatorname{Spec} A \sqcup \operatorname{Spec} B.$$

Proof. The algebraic structure of the product ring $R = A \times B$ is determined by the orthogonal idempotents $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Note that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$.

Any ideal in R is of the form $\mathfrak{a} \times \mathfrak{b}$, where $\mathfrak{a} \subseteq A$ and $\mathfrak{b} \subseteq B$ are ideals. A prime ideal $\mathfrak{p} \subset R$ must contain the product $e_1 e_2 = 0$. Since $\mathfrak{p}$ is prime, it must contain either e_1 or e_2.

If $e_2 \in \mathfrak{p}$, then $\mathfrak{p}$ corresponds to a prime ideal in the quotient ring $R/(e_2) \cong A$. Explicitly, such prime ideals are of the form $\mathfrak{p}_A \times B$, where $\mathfrak{p}_A \in \operatorname{Spec} A$. Geometrically, this set is exactly the closed set $V(e_2)$, which is homeomorphic to $\operatorname{Spec}(R/(e_2)) \cong \operatorname{Spec} A$. Furthermore, since $e_1 = 1 - e_2$, the condition $e_2 \in \mathfrak{p}$ is equivalent to $e_1 \notin \mathfrak{p}$, so this set is also the distinguished open set $D(e_1)$. Thus, $\operatorname{Spec} A$ is identified with a clopen (closed and open) subset of $\operatorname{Spec} R$.

Symmetrically, if $e_1 \in \mathfrak{p}$, the prime ideal is of the form $A \times \mathfrak{p}_B$ for $\mathfrak{p}_B \in \operatorname{Spec} B$. This corresponds to the clopen set $V(e_1) = D(e_2)$, which is homeomorphic to $\operatorname{Spec} B$.

Since every prime ideal contains exactly one of e_1, e_2 (it cannot contain both, as their sum is 1), $\operatorname{Spec} R$ is the disjoint union of these two clopen subsets. □

Proposition 13 Connectedness and Idempotents.

Let A be a ring. The topological space $\operatorname{Spec} A$ is **disconnected** if and only if A contains a non-trivial idempotent element e (i.e., $e^2 = e$ with $e \neq 0$ and $e \neq 1$).

Equivalently, $\operatorname{Spec} A$ is **connected** if and only if the only idempotents in A are 0 and 1.

Proof. First, assume A contains a non-trivial idempotent e. Set $f = 1 - e$. Then $e^2 = e$ implies $f^2 = (1 - e)^2 = 1 - 2e + e^2 = 1 - e = f$, so f is also a non-trivial idempotent.

Moreover, $ef = e(1-e) = e - e^2 = 0$ and $e + f = 1$. By the Chinese Remainder Theorem (or direct verification), the map $A \to A/(e) \times A/(f)$ is an isomorphism. By **Proposition 12**, $\operatorname{Spec} A \cong \operatorname{Spec}(A/(e)) \sqcup \operatorname{Spec}(A/(f))$. Since e and f are not units (otherwise they would be 1) and not zero, the quotient rings are non-zero, so their spectra are non-empty. Thus $\operatorname{Spec} A$ is the union of two disjoint non-empty open sets, meaning it is disconnected.

Conversely, assume $\operatorname{Spec} A$ is disconnected. Then we can write $\operatorname{Spec} A = X_1 \sqcup X_2$, where X_1 and X_2 are disjoint, non-empty closed sets. Since their union is the whole space, they are also open sets. Let $X_1 = V(I)$ and $X_2 = V(J)$ for ideals I, J. The condition $V(I) \cup V(J) = V(IJ) = \operatorname{Spec} A$ implies that every prime ideal contains IJ, so $IJ \subseteq \sqrt{(0)}$. The condition $V(I) \cap V(J) = V(I+J) = \emptyset$ implies that no prime ideal contains $I + J$, so $I + J = A$.

Since X_1 is both open and closed, and open sets in the Zariski topology are unions of basic open sets $D(g)$, and X_1 is quasi-compact (as a closed subset of a quasi-compact space), X_1 is a finite union of distinguished open sets. However, a more direct algebraic approach is to consider the structure sheaf (or simply the ring decomposition). Since X_1 and X_2 are disjoint clopen sets covering $\operatorname{Spec} A$, they correspond to a product decomposition of the ring A. Specifically, there exist comaximal ideals I, J with $IJ = 0$ (or nilpotent, which can be lifted to an idempotent).

More precisely, since $I + J = A$, we can write $1 = a + b$ with $a \in I, b \in J$. Since $X_1 \cap X_2 = \emptyset$, for any $\mathfrak{p} \in X_1$, $b \notin \mathfrak{p}$ (otherwise $1 \in \mathfrak{p}$), so b is a unit on X_1. Similarly a is a unit on X_2. This constructs a section of the structure sheaf that is 0 on X_2 and 1 on X_1. Since $A = \Gamma(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$, there exists a global element $e \in A$ such that $e(\mathfrak{p}) = 1$ for $\mathfrak{p} \in X_1$ and $e(\mathfrak{p}) = 0$ for $\mathfrak{p} \in X_2$. This implies $e(1-e)$ vanishes at all prime ideals, so it is nilpotent.

While lifting idempotents from modulo nilpotents is standard, for the purpose of this proof, we can rely on the fact that if $\operatorname{Spec} A$ is disconnected, there exists a decomposition $A \cong A_1 \times A_2$, which yields a non-trivial idempotent $(1, 0)$. □

Definition 14 Krull Dimension and Noetherian Spaces.

Let X be a topological space.

(1) The **Krull dimension** (or simply dimension) of X, denoted $\dim X$, is the supremum of the lengths n of chains of distinct irreducible closed subsets $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subseteq X$.

(2) X is called a **Noetherian topological space** if it satisfies the **descending chain condition (DCC)** on closed subsets: any sequence of closed sets $Z_1 \supseteq Z_2 \supseteq \ldots$ eventually stabilizes, i.e., there exists an integer k such that $Z_n = Z_k$ for all $n \geq k$.

Proposition 15 Decomposition of Noetherian Spaces.

Let X be a Noetherian topological space.

(1) X can be written as a finite union $X = X_1 \cup X_2 \cup \cdots \cup X_n$, where each X_i is an irreducible closed subset of X.

(2) If we require that no X_i is contained in another X_j (for $i \neq j$), then this decomposition is unique up to the reordering of the components. The sets X_i are called the **irreducible components** of X.

Proof. We first establish existence. Assume, for the sake of contradiction, that the collection Σ of closed subsets of X that cannot be written as a finite union of irreducible closed sets is non-empty. Since X is Noetherian, the collection Σ must contain a minimal element Z (by Zorn's Lemma applied to the complement, or the equivalent definition of Noetherian spaces: every non-empty family of closed sets has a minimal element). The set Z itself cannot be irreducible (otherwise it would not be in Σ). Thus, $Z = Z_1 \cup Z_2$ for some proper closed subsets $Z_1, Z_2 \subsetneq Z$. By the minimality of Z, both Z_1 and Z_2 can be written as finite unions of irreducible closed sets. Consequently, their union Z can also be written as such, contradicting the assumption that $Z \in \Sigma$. Thus, X itself admits such a decomposition.

Now we address uniqueness. Suppose $X = X_1 \cup \cdots \cup X_n = Y_1 \cup \cdots \cup Y_m$ are two irredundant decompositions into irreducible closed sets. For any X_i, we have $X_i = X_i \cap X = X_i \cap (\bigcup_j Y_j) = \bigcup_j (X_i \cap Y_j)$. Since X_i is irreducible, it must be contained in one of the sets in the union, say $X_i \subseteq Y_j$. By a symmetric argument, there exists some k such that $Y_j \subseteq X_k$. Thus $X_i \subseteq Y_j \subseteq X_k$. Since the decomposition is irredundant, we must have $i = k$, which implies $X_i = Y_j$. This process establishes a bijection between the sets $\{X_i\}$ and $\{Y_j\}$. □

Proposition 16 Noetherian Properties of the Spectrum.

Let A be a Noetherian ring and $X = \operatorname{Spec} A$.

(1) X is a Noetherian topological space.

(2) The irreducible components of X correspond bijectively to the minimal prime ideals of A. Specifically, if $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are the minimal primes, then $X = V(\mathfrak{p}_1) \cup \cdots \cup V(\mathfrak{p}_n)$ is the unique irreducible decomposition.

(3) Let $I \subseteq A$ be an ideal with a primary decomposition $I = \bigcap_{i=1}^m \mathfrak{q}_i$. This induces a decomposition $V(I) = \bigcup_{i=1}^m V(\sqrt{\mathfrak{q}_i})$. This decomposition corresponds to the irreducible decomposition of the closed set $V(I)$, where the isolated primes correspond to the irreducible components, and embedded primes correspond to redundant sub-varieties.

Proof. For assertion (1), recall that the closed sets of X are of the form $V(I)$. A descending chain of closed sets $V(I_1) \supseteq V(I_2) \supseteq \ldots$ corresponds to an ascending chain of radical ideals $\sqrt{I_1} \subseteq \sqrt{I_2} \subseteq \ldots$. Since A is a Noetherian ring, its ideals satisfy the ascending chain condition (ACC). Therefore, the sequence of radical ideals stabilizes, which implies the sequence of closed sets stabilizes. Thus X is a Noetherian space.

For assertion (2), we know from **Proposition 11** that irreducible closed subsets of X are of the form $V(\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Spec} A$. An irreducible component is a maximal irreducible closed subset. Since the map $\mathfrak{p} \mapsto V(\mathfrak{p})$ reverses inclusion ($V(\mathfrak{p}) \subseteq V(\mathfrak{q}) \iff \mathfrak{q} \subseteq \mathfrak{p}$), a maximal irreducible closed set corresponds to a minimal prime ideal. Since A is Noetherian, it has finitely many minimal prime ideals, ensuring the decomposition is finite.

For assertion (3), let $I = \bigcap_{i=1}^m \mathfrak{q}_i$ be a minimal primary decomposition, and let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ be the associated primes. The relation $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ generalizes to finite intersections, yielding

$$V(I) = V\left(\bigcap_{i=1}^{m} \mathfrak{q}_i\right) = \bigcup_{i=1}^{m} V(\mathfrak{q}_i) = \bigcup_{i=1}^{m} V(\mathfrak{p}_i).$$

The set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ is the set of associated primes of A/I. This set contains the minimal primes over I (isolated primes) and possibly embedded primes. If $\mathfrak{p}_j$ is an embedded prime, then $\mathfrak{p}_i \subsetneq \mathfrak{p}_j$ for some isolated prime $\mathfrak{p}_i$. Geometrically, this implies $V(\mathfrak{p}_j) \subsetneq V(\mathfrak{p}_i)$. Therefore, in the union $\bigcup V(\mathfrak{p}_i)$, the terms corresponding to embedded primes are contained in the terms corresponding to isolated primes. Removing these redundant terms leaves exactly the union over the minimal primes of A/I, which are the irreducible components of $V(I)$. Thus, primary decomposition provides the algebraic data for the geometric decomposition, with isolated components corresponding one-to-one to geometric components. □

Lemma 17 Topological Dimension and Ideal Height.

(1) Let X be an irreducible topological space and $U \subseteq X$ a non-empty open subset. Then U is dense in X and $\dim U = \dim X$.

(2) Let A be a ring and $X = \operatorname{Spec} A$. The topological dimension of X is equal to the Krull dimension of A, denoted $\dim A$. Furthermore, for any irreducible closed subset $Y = V(\mathfrak{p}) \subseteq X$, the codimension of Y in X corresponds to the height of the prime ideal $\mathfrak{p}$:

$$\operatorname{codim}(Y, X) = \operatorname{ht}(\mathfrak{p}).$$

Proof. For the first assertion, recall that in an irreducible space, every non-empty open set is dense. To compare the dimensions, consider a chain of irreducible closed subsets $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ in X. Since U is dense, the intersection $Z_i \cap U$ is non-empty for each i. Moreover, the intersection of an irreducible set with an open set is irreducible, so we obtain a sequence $Z_0 \cap U \subseteq Z_1 \cap U \subseteq \cdots \subseteq Z_n \cap U$ of irreducible closed subsets of U. If $Z_i \cap U = Z_{i+1} \cap U$, then taking closures in X would imply $Z_i = \overline{Z_i \cap U} = \overline{Z_{i+1} \cap U} = Z_{i+1}$, a contradiction. Thus the inclusions are strict, implying $\dim X \leq \dim U$. Conversely, let $W_0 \subsetneq \cdots \subsetneq W_n$ be a chain in U. Taking closures in X yields a chain $\overline{W_0} \subsetneq \cdots \subsetneq \overline{W_n}$ in X (strictness follows from intersecting back with U). Thus $\dim U \leq \dim X$, proving equality.

For the second assertion, the topological dimension of X is the supremum of lengths of chains of irreducible closed subsets. By the bijection between irreducible closed sets and prime ideals (reversing inclusion), a chain $Z_0 \subsetneq \cdots \subsetneq Z_n$ in X corresponds uniquely to a chain of prime ideals $\mathfrak{p}_n \subsetneq \cdots \subsetneq \mathfrak{p}_0$ in A. The supremum of the lengths of such chains of primes is precisely the definition of the Krull dimension of A. Similarly, the codimension of an irreducible closed subset Y in X is defined as the supremum of lengths of chains starting at Y, i.e., $Y = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subseteq X$. Under the Galois connection $Z \mapsto I(Z)$, this corresponds to a chain of prime ideals $\mathfrak{p}_n \subsetneq \cdots \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_0 = \mathfrak{p}$, where $\mathfrak{p}$ is the generic point of Y. This is exactly the definition of the height of the ideal $\mathfrak{p}$, denoted $\operatorname{ht}(\mathfrak{p})$. □

Proposition 18 Local Dimension and Specialization.

Let $X = \operatorname{Spec} A$. Let Y be an irreducible closed subset of X, and let $\mathfrak{p} \in \operatorname{Spec} A$ be the generic point of Y (so $Y = V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$). Then the following equalities hold:

$$\operatorname{codim}(Y, X) = \operatorname{codim}(V(\mathfrak{p}), X) = \operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p}A_{\mathfrak{p}}) = \dim(\operatorname{Spec} A_{\mathfrak{p}}) = \dim(A_{\mathfrak{p}}).$$

Proof. The first equality is simply the definition of Y as $V(\mathfrak{p})$. The second equality, $\operatorname{codim}(V(\mathfrak{p}), X) = \operatorname{ht}(\mathfrak{p})$, was established in **Lemma** 17.

For the third equality, consider the localization homomorphism $A \to A_{\mathfrak{p}}$. The correspondence theorem for localization states that there is a bijection between prime ideals of $A_{\mathfrak{p}}$ and prime ideals of A contained in $\mathfrak{p}$. Consequently, any chain of prime ideals $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$ in A extends uniquely to a chain $\mathfrak{p}_0 A_{\mathfrak{p}} \subsetneq \cdots \subsetneq \mathfrak{p}_n A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$ in $A_{\mathfrak{p}}$. Thus, the height of $\mathfrak{p}$ in A is identical to the height of the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$ in the local ring $A_{\mathfrak{p}}$.

For the fourth equality, recall that for any local ring $(R, \mathfrak{m})$, the dimension of the spectrum $\operatorname{Spec} R$ is determined by the maximum length of prime ideal chains ending at $\mathfrak{m}$. Since $\mathfrak{p}A_{\mathfrak{p}}$ is the unique maximal ideal of $A_{\mathfrak{p}}$, any maximal chain of primes in $A_{\mathfrak{p}}$ must end at $\mathfrak{p}A_{\mathfrak{p}}$. Therefore, the height of $\mathfrak{p}A_{\mathfrak{p}}$ is exactly the Krull dimension of the ring $A_{\mathfrak{p}}$, which is by definition $\dim(\operatorname{Spec} A_{\mathfrak{p}})$. Geometrically, this reflects that the local ring $A_{\mathfrak{p}}$ describes the "germ" of the space X near the point $\mathfrak{p}$, and the dimension of this local germ is the codimension of the closure $\overline{\{\mathfrak{p}\}}$ inside X. ◻

3.3 Affine Schemes

We now equip the topological space $X = \operatorname{Spec} A$ with a sheaf of rings, making it into a locally ringed space. Instead of using the cumbersome definition involving germs and compatible functions, we define the sheaf directly on the principal open sets and extend it using the Extension Theorem from Chapter 1.

Recall that the principal open sets $\{D(f) \mid f \in A\}$ form a base for the Zariski topology on X.

Definition 19 The Module on the Basis.

Let $\mathcal{B}$ denote the collection of principal open sets of X. We define an assignment $\widetilde{M}$ on $\mathcal{B}$ as follows:

1. *Sections:* For any $f \in A$, we define
$$\widetilde{M}(D(f)) := M_f = M \otimes_A A_f.$$
2. *Restrictions:* If $D(g) \subseteq D(f)$, then $g^n = uf$ for some $u \in A, n \geq 1$. This induces a canonical localization homomorphism $\rho_{g,f} : M_f \to M_g$, which acts as the restriction map.

To extend this to a sheaf on all of X, we first must verify that this assignment satisfies the sheaf axiom with respect to covers by basic open sets. This is the module-theoretic generalization of the partition of unity argument.

Lemma 20 Sheaf Property on the Basis.

Let $D(f)$ be a basic open set, and let $\{D(g_i)\}_{i\in I}$ be a cover of $D(f)$ by basic open sets. The following sequence of A-modules is exact:

$$0 \longrightarrow M_f \longrightarrow \prod_{i\in I} M_{g_i} \xrightarrow{d^0-d^1} \prod_{i,j\in I} M_{g_ig_j}$$

where the maps are the canonical restrictions (localizations).

Proof. Since $D(f)$ is quasi-compact, we can select a finite subcover $D(g_1),\dots,D(g_n)$ such that $D(f)=\bigcup_{k=1}^n D(g_k)$. If the sequence is exact for this finite subcover, it implies exactness for the arbitrary cover (since any element in the infinite product effectively lives on a finite refinement, and being zero is a local property). Thus, we assume $I=\{1,\dots,n\}$ is finite.

Step 1: Reduction to the case $f=1$. The modules $M_f, M_{g_i}, M_{g_ig_j}$ are all naturally modules over the localized ring A_f. The condition $D(f)=\bigcup D(g_i)$ implies that the elements g_i generate the unit ideal in A_f. By replacing A with A_f, M with M_f, and g_i with their images in A_f, we may assume without loss of generality that $f=1$ (so $D(f)=X$) and $(g_1,\dots,g_n)=A$. The sequence to prove becomes:

$$0 \longrightarrow M \xrightarrow{\alpha} \prod_{i=1}^n M_{g_i} \xrightarrow{\beta} \prod_{i,j} M_{g_ig_j}.$$

Step 2: Injectivity of α. Let $m\in M$ be an element such that $\alpha(m)=0$. This means the image of m in M_{g_i} is zero for all $i=1,\dots,n$. By the definition of localization, $m=0$ in M_{g_i} implies there exists an integer $k_i\geq 1$ such that $g_i^{k_i}m=0$ in M. Let $k=\max\{k_1,\dots,k_n\}$. Then $g_i^k m=0$ for all i. Since $(g_1,\dots,g_n)=A$, the ideal generated by powers $(g_1^k,\dots,g_n^k)$ is also the unit ideal A. Thus, there exist $a_1,\dots,a_n\in A$ such that:

$$\sum_{i=1}^n a_i g_i^k = 1.$$

Acting on m, we get:

$$m = 1\cdot m = \left(\sum_{i=1}^n a_i g_i^k\right) m = \sum_{i=1}^n a_i(g_i^k m) = \sum_{i=1}^n a_i\cdot 0 = 0.$$

Thus α is injective.

Step 3: Exactness at the middle (Gluing). Let $s=(s_1,\dots,s_n)\in\prod M_{g_i}$ be an element in the kernel of β. This means s_i and s_j map to the same element in $M_{g_ig_j}$ for all pairs i,j. We write $s_i=\frac{m_i}{g_i^{k_i}}$ for some $m_i\in M$ and $k_i\geq 0$. Since there are finitely many indices, we can pick a large integer N (larger than all k_i) such that we can write $s_i=\frac{m_i}{g_i^N}$ for all i (re-labeling the numerators m_i by multiplying with suitable powers of g_i).

The condition $s_i=s_j$ in $M_{g_ig_j}$ means:

$$\frac{m_i}{g_i^N}=\frac{m_j}{g_j^N} \quad \text{in } M_{g_ig_j}.$$

This implies there exists an integer $p_{ij} \geq 1$ such that:

$$(g_i g_j)^{p_{ij}}(g_j^N m_i - g_i^N m_j) = 0 \quad \text{in } M.$$

Since there are finitely many pairs, we can choose a single large integer P that works for all pairs. By replacing N with $N+P$ and replacing m_i with $m_i g_i^P$, we can assume the equality holds strictly in M without the auxiliary factor (or simply assume the factor is absorbed into the "large power" representation). Thus, we assume we have chosen N large enough such that for all i, j:

$$g_j^N m_i = g_i^N m_j \quad \text{in } M.$$

Since the ideal generated by $g_1, \ldots, g_n$ is A, the ideal generated by $g_1^N, \ldots, g_n^N$ is also A. So we can find coefficients $b_1, \ldots, b_n \in A$ such that:

$$\sum_{i=1}^{n} b_i g_i^N = 1.$$

We define a global element $x \in M$ by:

$$x := \sum_{j=1}^{n} b_j m_j.$$

We claim that x restricts to s_i on each $D(g_i)$. We compute the localization of x in M_{g_i}:

$$x|_{D(g_i)} = \frac{x}{1} = \frac{\sum_j b_j m_j}{1} = \frac{\sum_j b_j m_j \cdot g_i^N}{g_i^N}.$$

Using the compatibility relation $g_i^N m_j = g_j^N m_i$, we substitute into the sum:

$$\frac{\sum_j b_j (g_i^N m_j)}{g_i^N} = \frac{\sum_j b_j (g_j^N m_i)}{g_i^N} = \frac{(\sum_j b_j g_j^N) m_i}{g_i^N}.$$

Since $\sum b_j g_j^N = 1$, this simplifies to:

$$\frac{1 \cdot m_i}{g_i^N} = s_i.$$

Thus, x maps to $(s_i)_i$, proving surjectivity onto the kernel of β. □

We now define the sheaf on an arbitrary open set U using the "kernel definition" motivated by the sheaf axiom.

Definition 21 The Sheaf $\widetilde{M}$.

For any open set $U \subseteq X$, let $\mathcal{U} = \{D(f_i)\}_{i \in I}$ be a covering of U by basic open sets. We define $\widetilde{M}(U)$ to be the kernel of the difference map matching the sections on overlaps:

$$\widetilde{M}(U) := \ker\left(\prod_{i\in I}\widetilde{M}(D(f_i)) \xrightarrow{\rho_{ij}-\rho_{ji}} \prod_{i,j\in I}\widetilde{M}(D(f_i)\cap D(f_j))\right).$$

Here, strictly speaking, $D(f_i)\cap D(f_j) = D(f_i f_j)$ is also a basic open set, so the terms on the right are well-defined as $M_{f_i f_j}$.

Proposition 22 **Well-definedness of $\widetilde{M}(U)$.**

The definition of $\widetilde{M}(U)$ is independent of the choice of the basic covering $\mathcal{U}$, up to canonical isomorphism. Furthermore, $\widetilde{M}$ is a sheaf on X.

Proof. This follows from the fact that $\widetilde{M}$ is a sheaf on the basis.

If $\mathcal{V}$ is a refinement of $\mathcal{U}$, the restriction maps induce a morphism between the kernels. Since the sheaf property holds on basic opens, this morphism is an isomorphism.

For two arbitrary basic covers $\mathcal{U}$ and $\mathcal{W}$, one considers their common refinement $\mathcal{U}\cap\mathcal{W}$ to establish the isomorphism.

The global sheaf property follows formally because the condition is satisfied locally on the basis $\mathcal{B}$, and the definition for arbitrary U enforces the local-to-global gluing by construction. □

We now recover the standard structure sheaf as a specific instance of this construction.

Definition 23 **Quasi-coherent and Coherent Sheaves on Affine Schemes.**

On $X = \mathrm{Spec}(A)$, a sheaf $\mathcal{F}$ is quasi-coherent if and only if it is associated to some A-module M:

$$\mathcal{F} \cong \widetilde{M}$$

On $X = \mathrm{Spec}(A)$, if the ring A is Noetherian, then $\mathcal{F}$ is coherent if and only if it is associated to a finitely generated A-module M:

$$\mathcal{F} \cong \widetilde{M}, \quad M \in \text{mod-}A$$

Definition 24 **Structure Sheaf of Affine Schemes.**

Let A be a ring. The *structure sheaf* of $X = \mathrm{Spec}\,A$, denoted by $\mathcal{O}_X$, is defined as the sheaf associated to A viewed as a module over itself:

$$\mathcal{O}_X := \widetilde{A}.$$

Definition 25 **Affine Scheme.**

An *affine scheme* is a locally ringed space isomorphic to $(\mathrm{Spec}A, \mathcal{O}_{\mathrm{Spec}A})$ for some ring A. (Recall 26: The stalk $(\mathcal{O}_{\mathrm{Spec}A})_{\mathfrak{p}} \cong A_{\mathfrak{p}}$ is a local ring, so affine schemes are indeed locally ringed spaces).

By definition, for any principal open set $D(f)$, we have $\mathscr{O}_X(D(f)) = A_f$. In particular, the global sections are $\mathscr{O}_X(X) = A$.

We now prove the fundamental properties.

Proposition 26 Fundamental Properties of the Structure Sheaf.

Let $X = \operatorname{Spec} A$.

1. For any $f \in A$, $\Gamma(D(f), \mathscr{O}_X) \cong A_f$. In particular, $\Gamma(X, \mathscr{O}_X) \cong A$.
2. For any point $\mathfrak{p} \in X$, the stalk $\mathscr{O}_{X,\mathfrak{p}}$ is isomorphic to the local ring $A_\mathfrak{p}$.

Proof. *(i)* This is immediate from the construction. The Extension Theorem states that the extended sheaf agrees with the basis definition on basis sets. Thus $\mathscr{O}_X(D(f)) = A_f$.

(ii) By the definition of stalks, the stalk is the direct limit over open neighborhoods:

$$\mathscr{O}_{X,\mathfrak{p}} = \varinjlim_{U \ni \mathfrak{p}} \mathscr{O}_X(U).$$

Since the principal open sets $\{D(f) \mid f \notin \mathfrak{p}\}$ form a cofinal system of neighborhoods for $\mathfrak{p}$, we can compute the limit using only these sets:

$$\mathscr{O}_{X,\mathfrak{p}} \cong \varinjlim_{f \notin \mathfrak{p}} \mathscr{O}_X(D(f)) = \varinjlim_{f \notin \mathfrak{p}} A_f.$$

By the definition of localization at a prime ideal, the direct limit of A_f for all $f \notin \mathfrak{p}$ is precisely $A_\mathfrak{p}$. □

Intuition 27 Fundamental Properties of Sheaves of Modules.

We shall now proceed to demonstrate several key properties and isomorphisms concerning sheaves of modules (or the sheafification of A-modules). Although this will be quite involved and consume considerable space, we believe the effort is well-justified. It serves both as an excellent exercise in familiarizing oneself with the foundational tools of commutative algebra and sheaf theory, and as an opportune moment to observe the convergence of properties between sheaves of modules and sheaves of rings. Throughout the proofs, we will utilize several classical isomorphisms from commutative algebra, along with the commutation properties of the stalk functor.

Statement (i): $(\widetilde{M})_\mathfrak{p} \cong M_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} A$.

Proof. By the definition of the stalk of a sheaf, $(\widetilde{M})_\mathfrak{p}$ is the direct limit of sections over open neighborhoods of $\mathfrak{p}$. Since the principal open sets $D(f)$ form a basis for the topology of $\operatorname{Spec} A$, we have:

$$(\widetilde{M})_\mathfrak{p} = \varinjlim_{U \ni \mathfrak{p}} \widetilde{M}(U) = \varinjlim_{\mathfrak{p} \in D(f)} \widetilde{M}(D(f)).$$

Using property (ii) (or the construction of $\widetilde{M}$), we know $\widetilde{M}(D(f)) \cong M_f$. The condition

$\mathfrak{p} \in D(f)$ is equivalent to $f \notin \mathfrak{p}$. Thus:

$$(\widetilde{M})_{\mathfrak{p}} \cong \varinjlim_{f \notin \mathfrak{p}} M_f.$$

From commutative algebra, the direct limit of localizations M_f for $f \notin \mathfrak{p}$ is precisely the localization at the prime ideal, $M_{\mathfrak{p}}$. Therefore, $(\widetilde{M})_{\mathfrak{p}} \cong M_{\mathfrak{p}}$. □

Statement (ii): $\widetilde{M}(D(f)) \cong M_f$ for every $f \in A$. In particular, $\widetilde{M}(\operatorname{Spec} A) \cong M$. Also, $\widetilde{M}|_{D(f)} \cong \widetilde{M_f}$ via the identification $D(f) \cong \operatorname{Spec} A_f$.

Proof. The isomorphism $\widetilde{M}(D(f)) \cong M_f$ follows directly from the construction of the sheaf $\widetilde{M}$. Standard construction defines $\widetilde{M}$ such that it satisfies the sheaf axioms for the covering of $D(f)$ by standard open sets, yielding M_f as the space of sections. Setting $f = 1$, we get $\widetilde{M}(D(1)) = \widetilde{M}(\operatorname{Spec} A) \cong M_1 \cong M$.

For the second part, regarding the restriction: The topological space $D(f)$ is homeomorphic to $\operatorname{Spec} A_f$. For any prime $\mathfrak{q} \in D(f)$ (which corresponds to a prime in A_f), the stalk of $\widetilde{M}|_{D(f)}$ is $M_{\mathfrak{q}}$, which is isomorphic to $(M_f)_{\mathfrak{q}A_f}$. Since a sheaf is determined by its stalks, $\widetilde{M}|_{D(f)}$ is isomorphic to the sheaf associated to the A_f-module M_f, denoted as $\widetilde{M_f}$. □

Statement (iii): A sequence of A-modules $M' \to M \to M''$ is exact if and only if the sequence of $\mathscr{O}_{\operatorname{Spec} A}$-modules $\widetilde{M'} \to \widetilde{M} \to \widetilde{M''}$ is exact.

Proof. ($\Rightarrow$) Assume $M' \to M \to M''$ is exact. A sequence of sheaves is exact if and only if it is exact on stalks. For any $\mathfrak{p} \in \operatorname{Spec} A$, the sequence of stalks is:

$$(\widetilde{M'})_{\mathfrak{p}} \to (\widetilde{M})_{\mathfrak{p}} \to (\widetilde{M''})_{\mathfrak{p}}.$$

By (i), this is isomorphic to the sequence of localized modules:

$$M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}}.$$

Since localization is an exact functor (flatness of $A_{\mathfrak{p}}$ over A), the exactness of the original sequence implies the exactness of the localized sequence. Thus, the sheaf sequence is exact.

($\Leftarrow$) Assume $\widetilde{M'} \to \widetilde{M} \to \widetilde{M''}$ is exact. Then for every $\mathfrak{p} \in \operatorname{Spec} A$, the sequence of stalks $M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}}$ is exact. A fundamental property of modules is that a sequence is exact if and only if it is exact at every localization at a prime ideal (exactness is a local property). Therefore, $M' \to M \to M''$ is exact. □

Statement (iv): For a family M_i ($i \in I$), $\bigoplus \widetilde{M_i} \cong (\bigoplus M_i)^{\sim}$.

Proof. We verify the isomorphism on stalks. Let $\mathfrak{p} \in \operatorname{Spec} A$. The stalk of the direct sum of sheaves is the direct sum of their stalks:

$$\left(\bigoplus_{i\in I}\widetilde{M_i}\right)_{\mathfrak{p}} \cong \bigoplus_{i\in I}\left(\widetilde{M_i}\right)_{\mathfrak{p}}.$$

Using (i), this is $\bigoplus_{i\in I}(M_i)_{\mathfrak{p}}$. On the other hand, the stalk of the sheaf associated to the direct sum is:

$$\left(\left(\bigoplus_{i\in I}M_i\right)^{\sim}\right)_{\mathfrak{p}} \cong \left(\bigoplus_{i\in I}M_i\right)_{\mathfrak{p}}.$$

Since localization commutes with direct sums (i.e., $(\bigoplus N_i)_S \cong \bigoplus (N_i)_S$), we have:

$$\bigoplus_{i\in I}(M_i)_{\mathfrak{p}} \cong \left(\bigoplus_{i\in I}M_i\right)_{\mathfrak{p}}.$$

Since the sheaves have isomorphic stalks at every point, they are isomorphic. □

Statement (v): For a direct system $(M_i, \phi_{ij})_{i\in I}$, $\varinjlim \widetilde{M_i} \cong (\varinjlim M_i)^{\sim}$.

Proof. Similar to (iv), we check stalks at an arbitrary $\mathfrak{p} \in \operatorname{Spec} A$. Taking the direct limit of sheaves commutes with taking stalks:

$$\left(\varinjlim \widetilde{M_i}\right)_{\mathfrak{p}} \cong \varinjlim (\widetilde{M_i})_{\mathfrak{p}}.$$

By (i), this is $\varinjlim (M_i)_{\mathfrak{p}}$. For the right-hand side, the stalk is:

$$\left((\varinjlim M_i)^{\sim}\right)_{\mathfrak{p}} \cong \left(\varinjlim M_i\right)_{\mathfrak{p}}.$$

Since localization is a left adjoint functor (specifically, tensor product with $A_{\mathfrak{p}}$), it commutes with all colimits, including direct limits. Thus:

$$\varinjlim (M_i)_{\mathfrak{p}} \cong \left(\varinjlim M_i\right)_{\mathfrak{p}}.$$

The stalks are isomorphic everywhere, so the sheaves are isomorphic. □

Statement (vi): For any A-modules M and N, we have

$$\mathscr{H}om_{\mathscr{O}_{\operatorname{Spec} A}}(M^{\sim}, N^{\sim}) \cong \operatorname{Hom}_A(M, N),$$

$$M^{\sim} \otimes_{\mathscr{O}_{\operatorname{Spec} A}} N^{\sim} \cong (M \otimes_A N)^{\sim}.$$

If M is an A-module with finite presentation, then we have

$$\mathscr{H}om_{\mathscr{O}_{\operatorname{Spec} A}}(M^{\sim}, N^{\sim}) \cong (\operatorname{Hom}_A(M, N))^{\sim}.$$

Proof.

1. The Global Hom Isomorphism:

$$\mathrm{Hom}_{\mathrm{Spec}A}(\widetilde{M},\widetilde{N}) \cong \mathrm{Hom}_A(M,N)$$

We utilize the fundamental adjunction between the "associated sheaf" functor and the "global sections" functor. For any A-module M and any sheaf of $\mathscr{O}_{\mathrm{Spec}A}$-modules $\mathscr{F}$, there is a canonical bijection (adjointness):

$$\mathscr{H}om_{\mathscr{O}_{\mathrm{Spec}A}}(\widetilde{M},\mathscr{F}) \cong \mathrm{Hom}_A(M,\Gamma(\mathrm{Spec}A,\mathscr{F})).$$

This property states that $M \mapsto \widetilde{M}$ is the left adjoint to the global sections functor Γ. By applying this adjunction to the case where $\mathscr{F} = \widetilde{N}$, we get:

$$\mathscr{H}om_{\mathscr{O}_{\mathrm{Spec}A}}(\widetilde{M},\widetilde{N}) \cong \mathrm{Hom}_A(M,\Gamma(\mathrm{Spec}A,\widetilde{N})).$$

From part (ii) of the Proposition, we already know that $\Gamma(\mathrm{Spec}A,\widetilde{N}) \cong N$. Substituting this back yields the result immediately:

$$\mathscr{H}om_{\mathscr{O}_{\mathrm{Spec}A}}(\widetilde{M},\widetilde{N}) \cong \mathrm{Hom}_A(M,N).$$

2. The Tensor Product Isomorphism:

$$\widetilde{M} \otimes_{\mathscr{O}_{\mathrm{Spec}A}} \widetilde{N} \cong (M \otimes_A N)^{\sim}$$

Two sheaves on a scheme X are isomorphic if and only if their stalks at every point $p \in X$ are canonically isomorphic. We check the stalks at an arbitrary prime $\mathfrak{p} \in \mathrm{Spec}A$.

LHS Stalk: The stalk of a tensor product of sheaves is the tensor product of their stalks. Using part (i):

$$(\widetilde{M} \otimes_{\mathscr{O}_{\mathrm{Spec}A}} \widetilde{N})_{\mathfrak{p}} \cong (\widetilde{M})_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} (\widetilde{N})_{\mathfrak{p}} \cong M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}.$$

RHS Stalk: The stalk of the associated sheaf is the localization of the module. Using part (i):

$$((M \otimes_A N)^{\sim})_{\mathfrak{p}} \cong (M \otimes_A N)_{\mathfrak{p}}.$$

It is a standard fact in commutative algebra that localization commutes with tensor products, i.e., $(M \otimes_A N)_{\mathfrak{p}} \cong M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}$. Since the stalks are canonically isomorphic for all $\mathfrak{p}$, the sheaves are isomorphic.

3. The Sheaf Hom Isomorphism (under Finite Presentation):

$$\mathscr{H}om_{\mathscr{O}_{\mathrm{Spec}A}}(\widetilde{M},\widetilde{N}) \cong (\mathrm{Hom}_A(M,N))^{\sim}$$

Again, we verify the isomorphism by checking the stalks at an arbitrary $\mathfrak{p} \in \mathrm{Spec}A$.

LHS Stalk: As stated in the problem description (and standard sheaf theory), for M of finite presentation, the stalk of the sheaf Hom commutes with taking stalks:

$$(\mathscr{H}om_{\mathscr{O}_{\mathrm{Spec}A}}(\widetilde{M},\widetilde{N}))_{\mathfrak{p}} \cong \mathrm{Hom}_{A_{\mathfrak{p}}}((\widetilde{M})_{\mathfrak{p}},(\widetilde{N})_{\mathfrak{p}}) \cong \mathrm{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}}).$$

RHS Stalk: By part (i), the stalk is the localization of the Hom module:

$$((\mathrm{Hom}_A(M,N))^{\sim})_{\mathfrak{p}} \cong (\mathrm{Hom}_A(M,N))_{\mathfrak{p}}.$$

We require the algebraic isomorphism:

$$(\mathrm{Hom}_A(M,N))_{\mathfrak{p}} \cong \mathrm{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}}).$$

This is a known result in commutative algebra which holds true precisely when M is finitely presented. Since the stalks are isomorphic via this canonical map, the sheaves are isomorphic.

□

Proposition 28 Summary of Properties of Module Sheaves.

We summarize the following series of properties of module sheaves:

i. $(M^{\sim})_{\mathfrak{p}} \cong M_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec} A$.

ii. $M^{\sim}(D(f)) \cong M_f$ for every $f \in A$. In particular, taking $f = 1$, we get $M^{\sim}(\operatorname{Spec} A) \cong M$. Via the isomorphism $D(f) \cong \operatorname{Spec} A_f$, $M^{\sim}|_{D(f)}$ is identified with $M_f^{\sim}$.

iii. A sequence of A-modules

$$M' \longrightarrow M \longrightarrow M''$$

is exact if and only if the sequence of $\mathscr{O}_{\operatorname{Spec} A}$-modules

$$M'^{\sim} \longrightarrow M^{\sim} \longrightarrow M''^{\sim}$$

is exact.

iv. For any A-modules M and N, we have

$$\mathrm{Hom}_{\mathscr{O}_{\operatorname{Spec} A}}(M^{\sim},N^{\sim}) \cong \mathrm{Hom}_A(M,N),$$

$$M^{\sim} \otimes_{\mathscr{O}_{\operatorname{Spec} A}} N^{\sim} \cong (M \otimes_A N)^{\sim}.$$

If M is an A-module with finite presentation, then we have

$$\mathscr{H}om_{\mathscr{O}_{\operatorname{Spec} A}}(M^{\sim},N^{\sim}) \cong (\mathrm{Hom}_A(M,N))^{\sim}.$$

v. For a family M_i $(i \in I)$ of A-modules, we have

$$\bigoplus_i M_i^{\sim} \cong \left(\bigoplus_i M_i\right)^{\sim}.$$

vi. For a direct system $(M_i, \phi_{ij})_{i\in I}$ of A-modules, we have

$$\varinjlim_i M_i^{\sim} \cong \left(\varinjlim_i M_i\right)^{\sim}.$$

For the proof of item (vi), we directly appealed to the crucial fact that the sheafification functor $(\cdot)^{\sim} : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathscr{O}_{\operatorname{Spec} A}}$ is left adjoint to the global sections functor $\Gamma(\operatorname{Spec} A, \cdot) : \mathrm{Mod}_{\mathscr{O}_{\operatorname{Spec} A}} \to \mathrm{Mod}_A$. We will now provide the full proof of this adjunction.

Theorem 29 Adjunction Formula for Sheafification and Global Sections.

Let A be a ring and $X = \operatorname{Spec} A$. For any A-module M and any sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$, there is a natural bijection:

$$\Phi : \mathrm{Hom}_{\mathscr{O}_X}(\widetilde{M}, \mathscr{F}) \xrightarrow{\cong} \mathrm{Hom}_A(M, \Gamma(X, \mathscr{F})).$$

Proof. Let $\Gamma(X, \widetilde{M}) \cong M$ denote the canonical isomorphism identifying global sections of $\widetilde{M}$ with M.

1. *Construction of the map* Φ *(Left to Right)*

Given a morphism of sheaves $\phi : \widetilde{M} \to \mathscr{F}$, we induce a map on the global sections $\Gamma(X, \phi) : \Gamma(X, \widetilde{M}) \to \Gamma(X, \mathscr{F})$. Pre-composing this with the identification $M \cong \Gamma(X, \widetilde{M})$, we define $\Phi(\phi)$ as the composite:

$$M \xrightarrow{\cong} \Gamma(X, \widetilde{M}) \xrightarrow{\Gamma(X,\phi)} \Gamma(X, \mathscr{F}).$$

This is clearly an A-module homomorphism.

2. *Construction of the inverse map* Ψ *(Right to Left)*

Let $\psi : M \to \Gamma(X, \mathscr{F})$ be an A-module homomorphism. We need to construct a sheaf morphism $\widetilde{\psi} : \widetilde{M} \to \mathscr{F}$. Since the principal open sets $D(f)$ form a basis for the topology of X, it suffices to define the morphism on these open sets and ensure compatibility.

For any $f \in A$, consider the restriction map of the sheaf $\mathscr{F}$:

$$\mathrm{res}_{X,D(f)} : \Gamma(X, \mathscr{F}) \to \mathscr{F}(D(f)).$$

Composing ψ with this restriction gives an A-module homomorphism:

$$M \xrightarrow{\psi} \Gamma(X, \mathscr{F}) \xrightarrow{\mathrm{res}} \mathscr{F}(D(f)).$$

Notice that $\mathscr{F}(D(f))$ is naturally a module over $\mathscr{O}_X(D(f)) \cong A_f$.

3. *Key Step (Universal Property of Localization):*

The map $M \to \mathscr{F}(D(f))$ maps elements of M into an A_f-module. Since the target is an A_f-module, any $f \in A$ acts invertibly on the image. By the universal property of localization, this map extends *uniquely* to an A_f-linear map from the localization M_f:

$$\widetilde{\psi}_{D(f)} : M_f \to \mathscr{F}(D(f)).$$

Since $\widetilde{M}(D(f)) = M_f$ by definition, we have defined the required map of sections:

$$\widetilde{\psi}_{D(f)} : \widetilde{M}(D(f)) \to \mathscr{F}(D(f)).$$

4. *Verification* These local maps $\widetilde{\psi}_{D(f)}$ are compatible with restrictions $D(fg) \subseteq D(f)$ because of the uniqueness in the universal property of localization. Thus, they glue together to form a well-defined sheaf morphism $\widetilde{\psi} : \widetilde{M} \to \mathscr{F}$.

Finally, Φ and Ψ are mutually inverse:

i. Starting with $\psi : M \to \Gamma(X, \mathscr{F})$, constructing $\widetilde{\psi}$ and taking global sections (setting $f = 1$) recovers $\widetilde{\psi}_{D(1)} : M_1 \to \mathscr{F}(D(1))$, which is exactly ψ (since $M_1 = M$).

ii. Starting with $\phi : \widetilde{M} \to \mathscr{F}$, the induced map on localizations $M_f \to \mathscr{F}(D(f))$ is uniquely determined by its restriction to M by the universal property. Thus, reconstructing the sheaf map from the global map recovers ϕ.

Therefore, $\mathrm{Hom}_{\mathscr{O}_X}(\widetilde{M}, \mathscr{F}) \cong \mathrm{Hom}_A(M, \Gamma(X, \mathscr{F}))$. □

Using our basis-centric definitions and the established adjunctions, the proofs of standard compatibilities become trivial algebraic verifications.

Proposition 30 Compatibility with Modules.

Let $\phi : A \to B$ be a ring homomorphism and $f : \mathrm{Spec} B \to \mathrm{Spec} A$ the corresponding morphism of schemes.

(i) For any B-module N, there is a natural isomorphism $f_*(\widetilde{N}) \cong \widetilde{{}_A N}$, where ${}_A N$ denotes N regarded as an A-module.

(ii) For any A-module M, there is a natural isomorphism $f^*(\widetilde{M}) \cong \widetilde{B \otimes_A M}$.

Proof. Part (i): Direct Image.
Since the sheaves are defined on the basis of principal open sets, it suffices to verify the isomorphism on an arbitrary basic open set $D(a) \subseteq \mathrm{Spec} A$ for $a \in A$.

LHS: By definition of direct image and the associated sheaf:

$$(f_* \widetilde{N})(D(a)) = \widetilde{N}(f^{-1}(D(a))).$$

The preimage of a principal open set is principal: $f^{-1}(D(a)) = D(\phi(a)) \subseteq \mathrm{Spec} B$. Thus:

$$\widetilde{N}(D(\phi(a))) = N_{\phi(a)}.$$

RHS: By definition of the associated sheaf on $\mathrm{Spec} A$:

$$\widetilde{{}_A N}(D(a)) = ({}_A N)_a.$$

Algebraically, the localization of N at the element $\phi(a) \in B$ is canonically isomorphic to the localization of N (viewed as an A-module) at $a \in A$. The restriction maps are similarly compatible. Thus, $f_* \widetilde{N} \cong \widetilde{{}_A N}$.

Part (ii): Inverse Image.
We use the Uniqueness of Adjoints (Yoneda Perspective). We have established two adjunctions:

1. *Geometric:* $f^* \dashv f_*$. For any $\mathscr{O}_X$-module $\mathscr{F}$ and $\mathscr{O}_Y$-module $\mathscr{G}$:
$$\mathscr{H}om_{\mathscr{O}_Y}(f^*\mathscr{F}, \mathscr{G}) \cong \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, f_*\mathscr{G}).$$

2. *Algebraic:* $- \otimes_A B \dashv \mathrm{Res}^B_A$. For any A-module M and B-module N:
$$\mathrm{Hom}_B(M \otimes_A B, N) \cong \mathrm{Hom}_A(M, {}_AN).$$

Now, let $\mathscr{F} = \widetilde{M}$ and let $\mathscr{G} = \widetilde{N}$ be sheaves on $\mathrm{Spec}B$. We compute:

$$\begin{aligned}
\mathscr{H}om_{\mathscr{O}_{\mathrm{Spec}B}}(f^*\widetilde{M}, \widetilde{N}) &\cong \mathscr{H}om_{\mathscr{O}_{\mathrm{Spec}A}}(\widetilde{M}, f_*\widetilde{N}) \quad \text{(Geometric Adjunction)}\\
&\cong \mathscr{H}om_{\mathscr{O}_{\mathrm{Spec}A}}(\widetilde{M}, \widetilde{{}_AN}) \quad \text{(By Part (i))}\\
&\cong \mathrm{Hom}_A(M, {}_AN) \quad (\text{Full faithfulness of } M \mapsto \widetilde{M})\\
&\cong \mathrm{Hom}_B(M \otimes_A B, N) \quad \text{(Algebraic Adjunction)}\\
&\cong \mathscr{H}om_{\mathscr{O}_{\mathrm{Spec}B}}(\widetilde{M \otimes_A B}, \widetilde{N}) \quad \text{(Full faithfulness).}
\end{aligned}$$

Since this holds naturally for any $\widetilde{N}$, by the Yoneda lemma, we have:
$$f^*\widetilde{M} \cong \widetilde{M \otimes_A B}.$$

□

3.4 Category of Affine Schemes

Definition 31 Morphism of Schemes.

Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be locally ringed spaces (and in particular, schemes). A **morphism** of schemes is a pair $(f, f^\sharp)$, consisting of:

(i) A continuous map $f : X \to Y$;

(ii) A homomorphism of sheaves of rings $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ on Y.

This pair must satisfy the **local condition**: for every point $x \in X$, the induced homomorphism on the stalks
$$f^\sharp_x : \mathscr{O}_{Y,f(x)} \longrightarrow \mathscr{O}_{X,x}$$
is a **local homomorphism** of local rings. That is, if $\mathfrak{m}_{f(x)}$ and $\mathfrak{m}_x$ denote the unique maximal ideals of $\mathscr{O}_{Y,f(x)}$ and $\mathscr{O}_{X,x}$ respectively, then $(f^\sharp_x)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$.

Theorem 32 Equivalence of Categories.

Let A and B be rings. There is a natural bijection
$$\mathrm{Hom}_{\mathsf{Sch}}(\mathrm{Spec}B, \mathrm{Spec}A) \cong \mathrm{Hom}_{\mathsf{Ring}}(A, B).$$

Consequently, the category of affine schemes is equivalent to the opposite category of commutative rings with unity.

Proof. We construct the correspondence explicitly. Let $X = \mathrm{Spec} B$ and $Y = \mathrm{Spec} A$.

First, assume given a ring homomorphism $\phi : A \to B$. We construct a morphism of schemes $(f, f^\sharp) : X \to Y$. Define the continuous map f by contraction of prime ideals: $f(\mathfrak{q}) = \phi^{-1}(\mathfrak{q})$ for any $\mathfrak{q} \in \mathrm{Spec} B$. For the structure sheaf, observe that for any distinguished open set $D(g) \subseteq Y$ (where $g \in A$), the ring of sections is A_g. The map ϕ induces a homomorphism $A_g \to B_{\phi(g)}$, which is compatible with restrictions. Since distinguished open sets form a basis for the topology, this data glues uniquely to define a sheaf morphism $f^\sharp : \mathscr{O}_Y \to f_* \mathscr{O}_X$. On the stalks, for any $\mathfrak{q} \in X$ and $\mathfrak{p} = f(\mathfrak{q})$, the induced map is the localization $\phi_\mathfrak{q} : A_\mathfrak{p} \to B_\mathfrak{q}$. Since $\mathfrak{q}B_\mathfrak{q} \cap A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$, this is a local homomorphism. Thus $(f, f^\sharp)$ is a morphism of locally ringed spaces.

Conversely, let $(f, f^\sharp) : \mathrm{Spec} B \to \mathrm{Spec} A$ be a morphism of schemes. Taking global sections yields a ring homomorphism

$$\phi = \Gamma(f^\sharp) : \Gamma(Y, \mathscr{O}_Y) \to \Gamma(Y, f_* \mathscr{O}_X) = \Gamma(X, \mathscr{O}_X).$$

Since Y and X are affine, $\Gamma(Y, \mathscr{O}_Y) \cong A$ and $\Gamma(X, \mathscr{O}_X) \cong B$, so ϕ is a homomorphism $A \to B$.

The crucial step is to show that this ϕ reconstructs the original morphism $(f, f^\sharp)$. Let $g : X \to Y$ be the map induced by ϕ (i.e., $g(\mathfrak{q}) = \phi^{-1}(\mathfrak{q})$). We must show $f = g$. Let $\mathfrak{q} \in X$ be a point. The morphism condition gives a local homomorphism on stalks:

$$\theta : \mathscr{O}_{Y, f(\mathfrak{q})} \to \mathscr{O}_{X, \mathfrak{q}}.$$

Identifying stalks with localizations $A_{f(\mathfrak{q})}$ and $B_\mathfrak{q}$, we have a diagram commuting with the global map ϕ. The key property of a local homomorphism $\theta : A_\mathfrak{p} \to B_\mathfrak{q}$ is that $\theta^{-1}(\mathfrak{q}B_\mathfrak{q}) = \mathfrak{p}A_\mathfrak{p}$. In terms of the original rings, this means the preimage of $\mathfrak{q}$ under ϕ is exactly the prime ideal defining the stalk on the target, which is $f(\mathfrak{q})$. Therefore, $f(\mathfrak{q}) = \phi^{-1}(\mathfrak{q}) = g(\mathfrak{q})$.

Since f and g coincide topologically, and the sheaf morphisms coincide on global sections (by definition) and on stalks (by the local property essentially fixing the localization maps), the two morphisms are identical. □

Corollary 33 Affine Open Subschemes.

Let A be a ring and $f \in A$. The distinguished open set $D(f) \subseteq \mathrm{Spec} A$, equipped with the restriction of the structure sheaf $\mathscr{O}_{\mathrm{Spec} A}|_{D(f)}$, is an affine scheme isomorphic to $\mathrm{Spec} A_f$.

Proof. Consider the canonical localization homomorphism $\psi : A \to A_f$. By **Theorem** 32, this induces a morphism of schemes $\pi : \mathrm{Spec} A_f \to \mathrm{Spec} A$.

Topologically, we have already established (in Proposition 8) that the image of π is exactly $D(f)$ and that π is a homeomorphism onto $D(f)$. To show this is an isomorphism of schemes, we examine the structure sheaves. For any $g \in A$, basic open sets in $\mathrm{Spec} A_f$ are of the form $D(g/1)$ (viewing g inside A_f). The sections of $\mathscr{O}_{\mathrm{Spec} A_f}$ on such a set are $(A_f)_g \cong A_{fg}$. Similarly, the sections of $\mathscr{O}_{\mathrm{Spec} A}$ on the image $D(f) \cap D(g) = D(fg)$ are A_{fg}. The morphism $\pi^\sharp$ induces the identity map $A_{fg} \to A_{fg}$. Since the sheaf isomorphism holds on a basis of open sets, π is an isomorphism between $\mathrm{Spec} A_f$ and the open subscheme $(D(f), \mathcal{O}_{\mathrm{Spec} A}|_{D(f)})$. □

Definition 34 Affine Schemes over a Base.

Let $S = \mathrm{Spec} A$ be a fixed affine scheme.

(i) An **affine scheme over** S (or simply an affine S-scheme) is an affine scheme $X = \mathrm{Spec} B$ equipped with a morphism of schemes $\pi : X \to S$, called the **structure morphism**.

(ii) Algebraically, the structure morphism π corresponds to a ring homomorphism $\phi : A \to B$, which endows B with the structure of an A-algebra.

(iii) A **morphism of affine** S**-schemes** from (X, π_X) to (Y, π_Y) is a morphism of schemes $f : X \to Y$ such that the diagram commutes: $\pi_Y \circ f = \pi_X$.

The category of affine schemes over $S = \mathrm{Spec} A$ is denoted by $\mathsf{AffSch}_{/S}$.

Proposition 35 Duality of Fiber Product and Tensor Product.

The category $\mathsf{AffSch}_{/S}$ is equivalent to the opposite category of A-algebras, denoted $(A\text{-}\mathsf{Alg})^{\mathrm{op}}$.

Let $X = \mathrm{Spec} B$ and $Y = \mathrm{Spec} C$ be two affine schemes over $S = \mathrm{Spec} A$. The **fiber product** $X \times_S Y$ exists in the category of affine schemes and is given by the spectrum of the tensor product:

$$X \times_S Y \cong \mathrm{Spec}(B \otimes_A C).$$

The projection morphisms $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$ are induced by the canonical algebra homomorphisms $B \to B \otimes_A C$ $(b \mapsto b \otimes 1)$ and $C \to B \otimes_A C$ $(c \mapsto 1 \otimes c)$, respectively.

Proof. The equivalence of categories follows directly from **Theorem** 32. A morphism $X \to Y$ over S corresponds to a ring homomorphism $C \to B$ making the triangle with A commute, which is precisely the definition of an A-algebra homomorphism.

To prove the assertion about the fiber product, we verify the universal property. The fiber product $Z = X \times_S Y$ is defined as the limit of the diagram $X \to S \leftarrow Y$. Specifically, for any affine scheme $T = \mathrm{Spec} R$ over S, the set of S-morphisms $\mathrm{Hom}_S(T, Z)$ must be naturally bijective to the set of pairs of S-morphisms (f, g) where $f : T \to X$ and $g : T \to Y$.

Translating this via the contravariant functor $\mathrm{Spec}(\cdot)$, we seek a ring D such that

$$\mathrm{Hom}_{A\text{-Alg}}(D, R) \cong \mathrm{Hom}_{A\text{-Alg}}(B, R) \times \mathrm{Hom}_{A\text{-Alg}}(C, R).$$

This is precisely the universal property of the tensor product of A-algebras. The tensor product $B \otimes_A C$ is the coproduct in the category of A-algebras. Thus, for any A-algebra R, giving an A-algebra homomorphism $B \otimes_A C \to R$ is equivalent to giving a pair of A-algebra homomorphisms $B \to R$ and $C \to R$.

Applying the functor Spec back to the geometric category, we conclude that $\mathrm{Spec}(B \otimes_A C)$, equipped with the morphisms induced by the canonical injections of B and C into the tensor product, satisfies the universal property of the fiber product $X \times_S Y$. □

Intuition 36 .

1. **Intersection of Equations:** If X and Y are closed subschemes of affine space $\mathbb{A}^n_A$, defined by ideals I and J in $A[x_1, \ldots, x_n]$, their fiber product over S corresponds to the geometric intersection. Algebraically, $A[x]/I \otimes_{A[x]} A[x]/J \cong A[x]/(I + J)$, which represents the union of equations.
2. **Base Change:** If $Y = \mathrm{Spec}A'$ and morphism $Y \to S$ represents a field extension or a localization, then $X \times_S Y$ represents the scheme X with its coefficients extended or restricted to the new base A'. For example, $\mathrm{Spec}\mathbb{R}[x]/(x^2 + 1) \times_{\mathrm{Spec}\mathbb{R}} \mathrm{Spec}\mathbb{C} \cong \mathrm{Spec}(\mathbb{C}[x]/(x^2 + 1)) \cong \mathrm{Spec}(\mathbb{C} \times \mathbb{C})$, showing how the irreducible polynomial splits after base change.

3.5 Examples

Example 37 The Prime Spectrum of the Integers.

The affine scheme $X = \mathrm{Spec}\mathbb{Z}$ is the fundamental object of arithmetic geometry. Its structure encodes the basic properties of prime numbers and the rational field within the language of schemes.

Set-Theoretic and Topological Structure. The points of X are the prime ideals of the ring $\mathbb{Z}$. These fall into two distinct classes:

(i) The **closed points**, corresponding to the maximal ideals generated by prime numbers, written as $(2), (3), (5), \ldots$. The residue field at a closed point $x = (p)$ is the finite field $\kappa(x) = \mathbb{Z}/(p) \cong \mathbb{F}_p$.

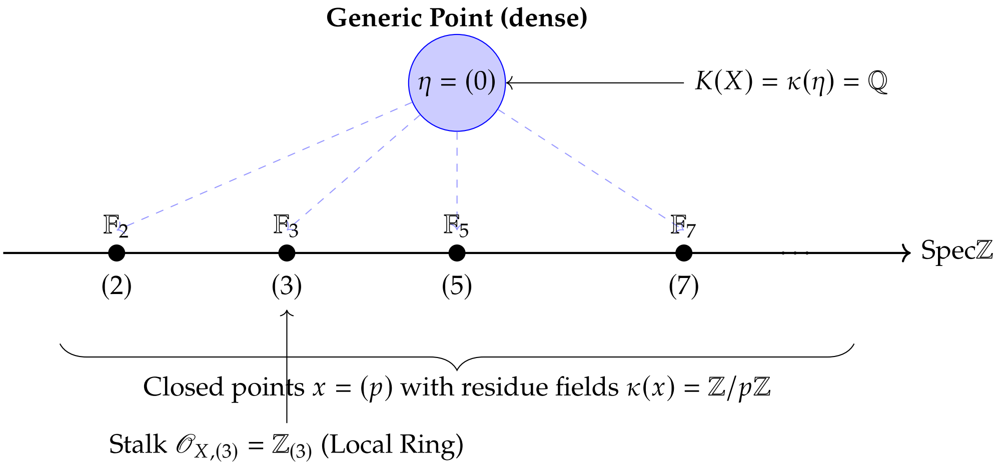
(ii) The **generic point** $\eta = (0)$, corresponding to the zero ideal. Since (0) is contained in every other prime ideal, the closure of η is the entire space X. Thus, η is a generic point in the topological sense, and X is an irreducible topological space.

The topology on X is the **cofinite topology** on the set of closed points, augmented by the generic point. Specifically, the closed subsets are X itself and finite unions of closed points. This arises because $\mathbb{Z}$ is a Principal Ideal Domain (PID) of Krull dimension 1; any non-zero ideal $I = (n)$ has a prime factorization $n = p_1^{e_1} \ldots p_k^{e_k}$, implying $V(I) = \{(p_1), \ldots, (p_k)\}$. Consequently, the non-empty open sets are the complements of finite sets of closed points.

Sheaf Structure and Stalks. The structure sheaf $\mathcal{O}_X$ is determined by the localizations of $\mathbb{Z}$.

- **Global Sections:** The ring of global sections is $\Gamma(X, \mathcal{O}_X) \cong \mathbb{Z}$. This reflects that holomorphic functions defined everywhere on the "arithmetic line" are just the integers.
- **Basic Open Sets:** For any non-zero integer f, the principal open set $D(f)$ consists of all primes not dividing f. The ring of sections over $D(f)$ is the localization $\mathbb{Z}_f = \mathbb{Z}[1/f]$. For example, on $D(2)$, the sections are rational numbers with denominators being powers of 2.

- **Stalks:** The stalk $\mathcal{O}_{X,(p)}$ at a closed point (p) is the discrete valuation ring $\mathbb{Z}_{(p)}$, consisting of rational numbers a/b where $p \nmid b$. This local ring captures the arithmetic behavior "near" the prime p. The stalk at the generic point η is the localization of $\mathbb{Z}$ at (0), which is the field of rational numbers $\mathbb{Q}$. This aligns with the function field of the scheme, $K(X) = \mathbb{Q}$.



Categorical Property. The scheme Spec$\mathbb{Z}$ is the **terminal object** in the category of affine schemes, and indeed in the category of all schemes. For any scheme Y, there exists a unique morphism $Y \to \operatorname{Spec}\mathbb{Z}$. Algebraically, this is dual to the fact that $\mathbb{Z}$ is the **initial object** in the category of rings: for any ring A, there is a unique ring homomorphism $\chi : \mathbb{Z} \to A$ (determined by $\chi(1) = 1_A$). Geometrically, this means that every scheme can be viewed as a family of schemes over Spec$\mathbb{Z}$. The fiber of a morphism $Y \to \operatorname{Spec}\mathbb{Z}$ over a point (p) is the scheme $Y_p = Y \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_p$, which is a scheme over characteristic p, while the fiber over (0) is $Y_{\mathbb{Q}} = Y \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Q}$, a scheme over characteristic 0. Thus, Spec$\mathbb{Z}$ unifies geometry across all characteristics.

Example 38 The Affine Line over $\mathbb{C}$.

Let $k = \mathbb{C}$ be the field of complex numbers. The **affine line** over $\mathbb{C}$, denoted by $\mathbb{A}^1_{\mathbb{C}}$, is defined as the spectrum of the polynomial ring in one variable:

$$X = \operatorname{Spec}\mathbb{C}[t].$$

Since $\mathbb{C}[t]$ is an integral domain, X is an **integral scheme**.

Points and Topological Structure. The ring $\mathbb{C}[t]$ is a Principal Ideal Domain (PID). Its prime ideals fall into two categories, mirroring the structure of Spec$\mathbb{Z}$:

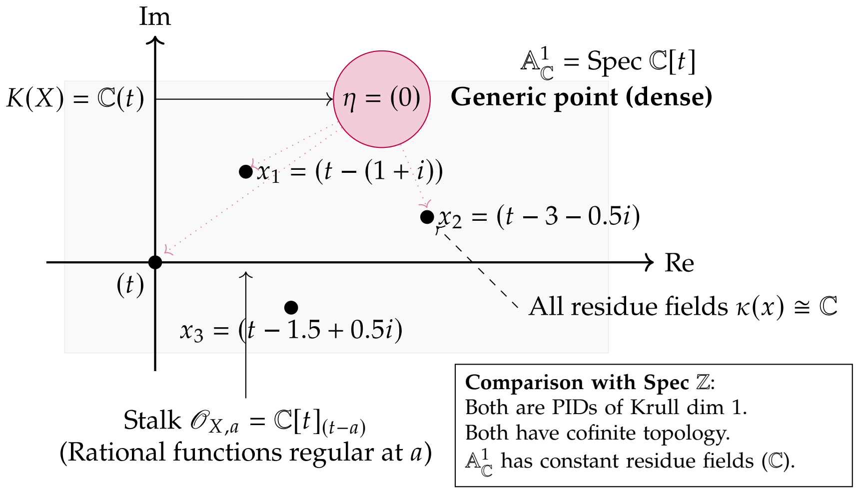
(i) **Closed Points:** By Hilbert's Nullstellensatz, the maximal ideals of $\mathbb{C}[t]$ are of the form $(t - a)$ for some $a \in \mathbb{C}$. Thus, there is a natural bijection between the closed points of X and the classical points of the complex line $\mathbb{C}$. The residue field at a closed point $x = (t - a)$ is $\mathbb{C}[t]/(t - a) \cong \mathbb{C}$.

(ii) **Generic Point:** The zero ideal (0) is prime but not maximal. The point $\eta = (0)$ is dense in X, as its closure is $V(0) = X$.

The topology is the **cofinite topology** (on the set of closed points). A proper closed set $V(I)$ is defined by a non-zero ideal $I = (f(t))$. Since $f(t)$ has finitely many roots, $V(I)$ consists of a finite number of closed points. Consequently, non-empty open sets are complements of finite sets. The dimension of the space is $\dim X = \dim \mathbb{C}[t] = 1$.

Sheaf Structure and Stalks.

- **Global Sections:** $\Gamma(X, \mathscr{O}_X) = \mathbb{C}[t]$. The regular functions on the entire affine line are simply polynomials.
- **Local Sections:** For a basic open set $D(f)$ where $f \in \mathbb{C}[t]$, the ring of sections is the localization $\mathbb{C}[t]_f$. Elements are rational functions $g(t)/f(t)^n$ which are regular away from the roots of f.
- **Stalks:** The stalk at a closed point a is the local ring $\mathbb{C}[t]_{(t-a)}$. This consists of rational functions $p(t)/q(t)$ where $q(a) \neq 0$ (regular at a). The stalk at the generic point η is the function field $K(X) = \mathbb{C}(t)$, the field of rational functions.



Comparison with Spec $\mathbb{Z}$:
Both are PIDs of Krull dim 1.
Both have cofinite topology.
$\mathbb{A}^1_{\mathbb{C}}$ has constant residue fields ($\mathbb{C}$).

Comparison with Spec$\mathbb{Z}$. The schemes Spec$\mathbb{Z}$ and $\mathbb{A}^1_{\mathbb{C}}$ are strikingly similar ("brothers" in the world of schemes):

(a) **Algebraic Structure:** Both underlying rings ($\mathbb{Z}$ and $\mathbb{C}[t]$) are Principal Ideal Domains (PIDs) of Krull dimension 1. This gives them an almost identical topological structure (cofinite topology + generic point).

(b) **Function Theory:** In both cases, "functions" behave like rational quantities. In Spec$\mathbb{Z}$, a rational number a/b is "regular" at a prime p if $p \nmid b$. In $\mathbb{A}^1_{\mathbb{C}}$, a rational function f/g is regular at a if $g(a) \neq 0$.

(c) **Difference (Residue Fields):** The key difference lies in the residue fields. For $\mathbb{A}^1_{\mathbb{C}}$, every closed point has the same residue field $\mathbb{C}$ (algebraically closed). For Spec$\mathbb{Z}$, the residue field at (p) is $\mathbb{F}_p$, which varies with the point and is finite.

Geometrically, Spec$\mathbb{Z}$ is often visualized as a "curve," just like the complex line.

Example 39 Affine n-Space over an Algebraically Closed Field.

Let k be an algebraically closed field (e.g., $k = \mathbb{C}$). The **affine n-space**, denoted by $\mathbb{A}^n_k$, is defined as the prime spectrum of the polynomial ring in n variables:

$$\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n].$$

Since the polynomial ring $R = k[x_1, \ldots, x_n]$ is an integral domain, $\mathbb{A}^n_k$ is an **integral scheme**, irreducible and reduced.

Topological Structure and Nullstellensatz. The space is equipped with the Zariski topology, where closed sets are of the form $V(I)$ for ideals $I \subseteq R$. Since R is a Noetherian ring (by Hilbert's Basis Theorem), $\mathbb{A}^n_k$ is a **Noetherian topological space**, meaning every descending chain of closed subsets stabilizes. The points of $\mathbb{A}^n_k$ correspond to prime ideals $\mathfrak{p} \subset R$. Due to the assumption that k is algebraically closed, Hilbert's Nullstellensatz provides a precise classification of these points:

(i) **Closed Points:** The maximal ideals are of the form $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ for $(a_1, \ldots, a_n) \in k^n$. Thus, the closed points of the scheme are in natural bijection with the points of the classical vector space k^n. The residue field at any closed point is $R/\mathfrak{m} \cong k$.

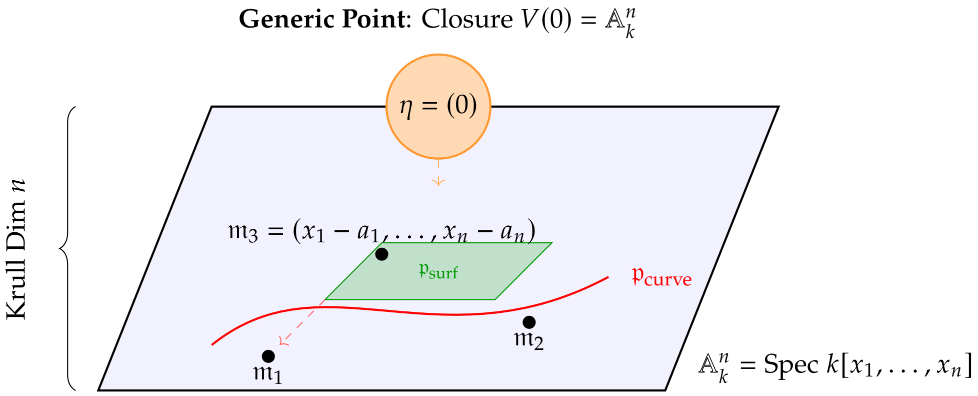
(ii) **Non-closed Points:** These correspond to non-maximal prime ideals. They represent irreducible closed subvarieties of dimension greater than 0 (e.g., curves, surfaces).

(iii) **Generic Point:** The zero ideal (0) is prime and corresponds to the generic point η. Its closure is the entire space $V(0) = \mathbb{A}^n_k$, reflecting that the space is irreducible.

The Krull dimension of $\mathbb{A}^n_k$ is equal to the Krull dimension of R, which is n.

Sheaf of Rings, Sections, and Stalks. The structure sheaf $\mathscr{O}_{\mathbb{A}^n_k}$ encapsulates the algebraic functions on the space.

- **Global Sections:** $\Gamma(\mathbb{A}^n_k, \mathscr{O}) \cong k[x_1, \ldots, x_n]$. This signifies that any function regular everywhere on affine space must be a polynomial.
- **Local Sections:** On a basic open set $D(f)$, sections are elements of the localization R_f, i.e., rational functions whose denominators are powers of f.
- **Stalks:** For a closed point $x = (a_1, \ldots, a_n)$, the stalk $\mathscr{O}_{\mathbb{A}^n_k, x}$ is the local ring $R_{(x_1 - a_1, \ldots, x_n - a_n)}$, consisting of rational functions P/Q where $Q(a) \neq 0$. The stalk at the generic point η is the field of rational functions $K(\mathbb{A}^n_k) = k(x_1, \ldots, x_n)$.



Product Formula. A fundamental property of affine space is the isomorphism

$$\mathbb{A}_k^n \times_{\operatorname{Spec} k} \mathbb{A}_k^m \cong \mathbb{A}_k^{n+m}.$$

To prove this, recall that the fiber product of affine schemes corresponds to the tensor product of their coordinate rings over the base ring. Thus,

$$\operatorname{Spec} k[x_1, \dots, x_n] \times_{\operatorname{Spec} k} \operatorname{Spec} k[y_1, \dots, y_m] \cong \operatorname{Spec}(k[x_1, \dots, x_n] \otimes_k k[y_1, \dots, y_m]).$$

Since the tensor product of polynomial rings is canonically isomorphic to the polynomial ring in the union of variables, $k[\bar{x}] \otimes_k k[\bar{y}] \cong k[x_1, \dots, x_n, y_1, \dots, y_m]$, the result follows immediately.

Remark on the Base Field. The properties described above rely primarily on k being an algebraically closed field. If we replace $\mathbb{C}$ with any other algebraically closed field (such as $\overline{\mathbb{F}}_p$), the topological structure (Nullstellensatz), dimension theory, and sheaf properties remain identical. The only variation arises in the arithmetic of the coefficients, but the geometric structure of the scheme is invariant.

Example 40 Affine Algebraic Sets and Varieties.

Let k be a field and let $R = k[x_1, \dots, x_n]$ be the polynomial ring in n variables.

1. Affine Algebraic Sets (Scheme-Theoretic View) Let I be an arbitrary ideal of R. The affine scheme defined by

$$X = \operatorname{Spec}(R/I)$$

is called an **affine algebraic set** (or an affine scheme of finite type over k). By the closed immersion property (**Proposition** 8), X is canonically homeomorphic to the closed subset $V(I) \subseteq \mathbb{A}_k^n$. The ring $A = R/I$ is called the **coordinate ring** of X, denoted by $A(X)$. Note that if I contains nilpotent elements (i.e., $I \neq \sqrt{I}$), the scheme X contains "infinitesimal thickening" information, which has no classical analogue.

2. Affine Varieties Now, impose the stronger assumptions that:

(i) The base field k is **algebraically closed**.

(ii) The ideal $I = \mathfrak{p}$ is a **prime ideal**.

Under these conditions, $X = \operatorname{Spec}(R/\mathfrak{p})$ is called an **affine variety**. The coordinate ring $A(X) = R/\mathfrak{p}$ is a finitely generated k-algebra which is also an **integral domain**. Consequently, X is an **integral scheme** (it is both reduced and irreducible).

Classical vs. Scheme Correspondence For an affine variety $X = \operatorname{Spec}(R/\mathfrak{p})$:

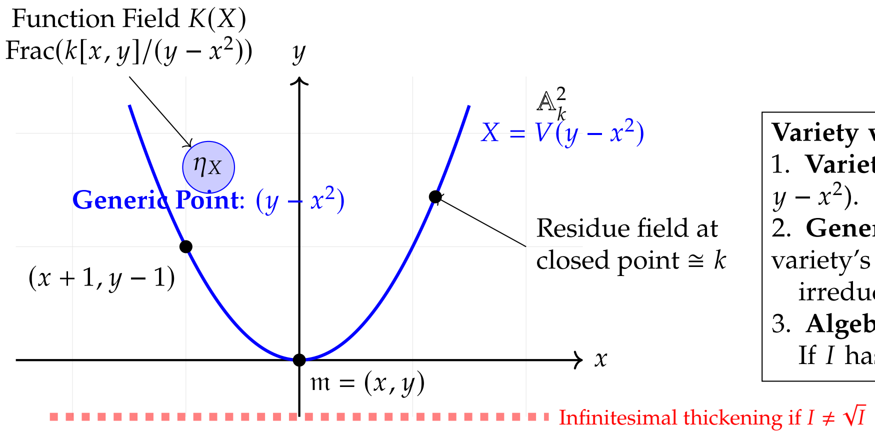
- **Closed Points:** By the Nullstellensatz, the maximal ideals of $A(X)$ correspond one-to-one with the points of the classical zero locus

  $$Z(\mathfrak{p}) = \{a \in k^n \mid f(a) = 0 \text{ for all } f \in \mathfrak{p}\}.$$

  Thus, the set of closed points recovers the classical geometric object.

- **Generic Point:** Since $\mathfrak{p}$ is prime, the zero ideal (0) in $A(X)$ is prime. This corresponds to the generic point η of X. Its closure is the entire variety X.

- **Function Field:** The stalk at the generic point, $\mathcal{O}_{X,\eta}$, is the fraction field of the coordinate ring, denoted $K(X) = \mathrm{Frac}(A(X))$. This is the **function field** of the variety, consisting of rational functions defined on dense open sets.



Variety vs. Algebraic Set:
1. **Variety**: $I = \mathfrak{p}$ is prime (e.g., $y - x^2$).
2. **Generic Point**: η_X tracks the variety's irreducible nature.
3. **Algebraic Set**: I is any ideal. If I has nilpotents, X has "fuzz".

Summary: An affine variety is the scheme-theoretic enrichment of a classical irreducible algebraic set, adding a generic point to track generic properties and a structure sheaf to track regular functions intrinsically.

Example 41 The Fat Point: Schemes with Nilpotents.

Let k be a field and consider the ring $A = k[x]/(x^n)$ for some integer $n \geq 2$. Let $X = \mathrm{Spec} A$. This object is often loosely referred to as a "fat point" of order n.

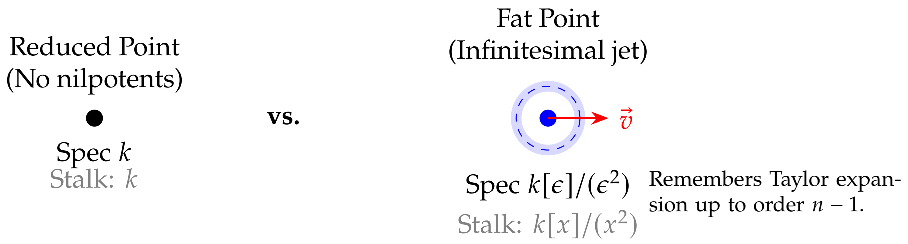
Topological Structure vs. Algebraic Structure. The only prime ideal in $k[x]$ containing (x^n) is (x). Consequently, the spectrum X consists of a **single point**, corresponding to the maximal ideal $\mathfrak{m} = (x) \subset A$. Topologically, X is a one-point space equipped with the discrete topology. In this sense, it is homeomorphic to Spec$k$ (the standard geometric point).

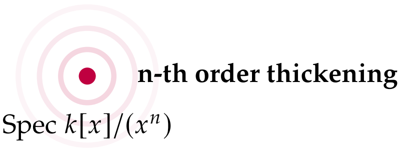
$$\mathrm{Spec}(k[x]/(x^n)) \approx_{\text{top}} \mathrm{Spec} k.$$

However, as schemes, they are strictly not isomorphic. The structure sheaf $\mathcal{O}_X$ at the unique point has the stalk $A = k[x]/(x^n)$, which contains non-zero nilpotent elements (specifically, the class of x). In contrast, the stalk of Spec$k$ is the field k, which is reduced.

Geometric Intuition: Infinitesimals. The scheme X should be visualized as a point equipped with an "infinitesimal neighborhood."

- Case $n = 2$: Spec$k[\epsilon]/(\epsilon^2)$ is the **algebra of dual numbers**. A morphism from this scheme to any other scheme Y is equivalent to specifying a point $y \in Y$ and a tangent vector at y. Thus, this "point" carries tangent direction data.
- General n: It represents the n-th order infinitesimal jet at the origin.





Failure of Classical/Analytic Geometry. In classical algebraic geometry or standard analytic geometry, spaces are usually sets of points defined by the vanishing of functions. The set of points in the affine line $\mathbb{A}^1_k$ where $x^n = 0$ is simply $\{0\}$.

- A classical function f is determined by its values at points. On $\{0\}$, the functions $f(x) = 0$ and $g(x) = x$ take the same value (0). Classical geometry cannot distinguish them.
- In scheme theory, f and g are distinct elements of the ring $k[x]/(x^n)$. The scheme remembers not just the value of the function (0), but its Taylor expansion up to order $n-1$.

Therefore, scheme theory provides the unique ability to study **infinitesimal deformations** and multiplicities purely algebraically, a feature completely absent in the set-theoretic or reduced analytic viewpoints.

Example 42 Zero-Dimensional Scheme with Nilpotents.

Consider the scheme $X = \operatorname{Spec} A$ where $A = \mathbb{C}[x]/(x^3 - x^2)$.

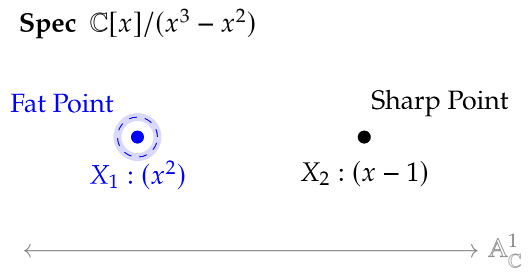
Algebraic Decomposition: The defining polynomial factors as $x^2(x-1)$. The ideals (x^2) and $(x-1)$ are comaximal in $\mathbb{C}[x]$ (since $-(x+1)(x-1) + 1 \cdot x^2 = 1$). By the Chinese Remainder Theorem, we have a ring isomorphism:

$$A \cong \mathbb{C}[x]/(x^2) \times \mathbb{C}[x]/(x-1).$$

Geometric Structure: According to the property of the spectrum of a direct product, X is the disjoint union of two schemes:

$$X \cong \operatorname{Spec}(\mathbb{C}[x]/(x^2)) \sqcup \operatorname{Spec}(\mathbb{C}[x]/(x-1)).$$

- The first component $X_1 = \operatorname{Spec} \mathbb{C}[x]/(x^2)$ is the "fat point" (or double point) at the origin. It has one underlying topological point, but its structure sheaf carries a tangent vector (nilpotent element of order 2).
- The second component $X_2 = \operatorname{Spec} \mathbb{C} \cong \operatorname{Spec} \mathbb{C}[x]/(x-1)$ is a standard "reduced" geometric point at $x = 1$.



Conclusion: X consists of two points. One is "fuzzy" (non-reduced), and the other is "sharp" (reduced). They are topologically disconnected. This illustrates how algebraic multiplicities translate into the scheme-theoretic structure.

Example 43 The Hyperbola (Localization).

Consider the affine scheme $X = \operatorname{Spec} k[x, y]/(xy - 1)$.

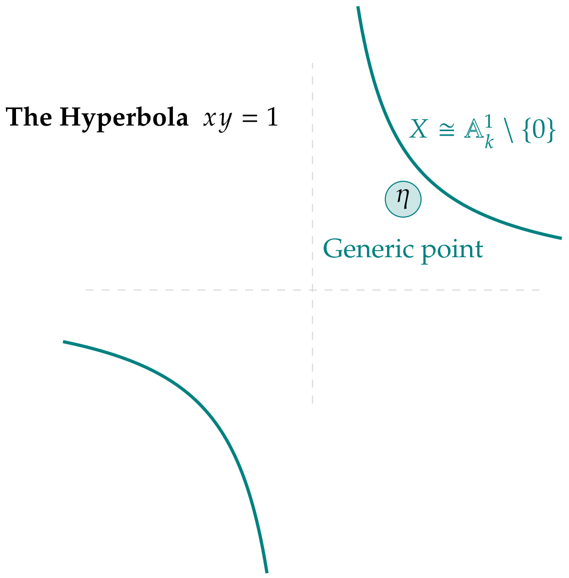
Identification as Localization: In the ring $A = k[x, y]/(xy - 1)$, the relation $xy = 1$ implies that x is a unit (with inverse y). Thus, the ring A is isomorphic to the localization of the polynomial ring in one variable at x:

$$A \cong k[x, x^{-1}] \cong k[x]_x.$$

Geometric Interpretation: Since $A \cong k[x]_x$, the scheme X is isomorphic to the distinguished open set $D(x)$ inside the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[x]$.

$$X \cong \mathbb{A}^1_k \setminus \{(0)\}.$$

Geometrically, this is the affine line with the origin removed. In the plane $\mathbb{A}^2_k$, this is the hyperbola. The projection map $\pi : X \to \mathbb{A}^1_k$ given by $(x, y) \mapsto x$ is an isomorphism onto $\mathbb{A}^1_k \setminus \{0\}$.



Properties:

- X is an affine variety (integral, reduced, irreducible).

- X is a "group scheme" (the multiplicative group $\mathbb{G}_m$), where the group law corresponds to the comultiplication $A \to A \otimes A$ defined by $x \mapsto x \otimes x$.

Example 44 The Axes Cross (Reducible Scheme).

Consider the scheme $X = \mathrm{Spec} k[x, y]/(xy)$.

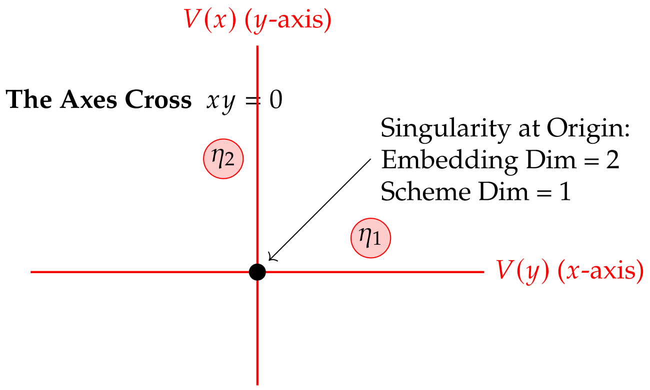
Reducibility: The ideal (xy) is not prime because $x \notin (xy)$ and $y \notin (xy)$, but their product is. Thus, X is not an integral scheme. The ideal decomposes as an intersection of two prime ideals:

$$(xy) = (x) \cap (y).$$

Geometrically, this corresponds to the union of closed subsets:

$$X = V(x) \cup V(y).$$

Here, $V(x)$ corresponds to the y-axis (where $x = 0$) and $V(y)$ corresponds to the x-axis. Thus, X is the union of two copies of the affine line $\mathbb{A}^1_k$ intersecting at the origin.



Generic Points: Unlike an integral scheme, X has **two** generic points, corresponding to the minimal prime ideals (x) and (y). The closure of (x) is the y-axis, and the closure of (y) is the x-axis.

Singularity at the Origin: At any point p on the axes other than the origin, the local ring is a discrete valuation ring (isomorphic to $k[t]_{(t)}$). However, at the origin $O = (x, y)$, the local ring is:

$$\mathcal{O}_{X,O} \cong (k[x, y]/(xy))_{(x,y)}.$$

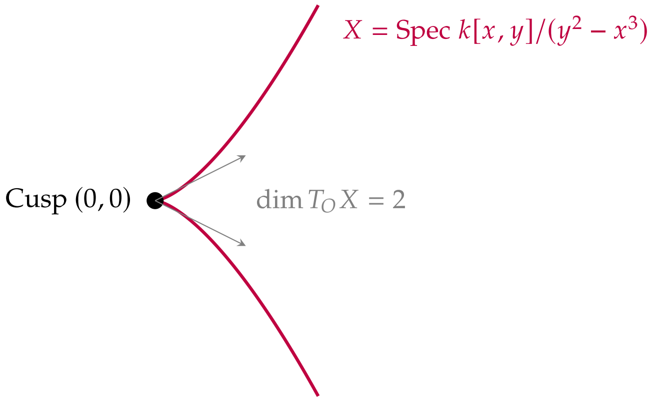
This is not an integral domain (since $xy = 0$ still holds). The tangent space dimension at the origin is 2 (embedding dimension), while the dimension of the scheme is 1. This algebraic fact detects the geometric "crossing" singularity.

Example 45 The Cuspidal Cubic (Singular Curve).

Consider the scheme $X = \mathrm{Spec} A$ with $A = k[x, y]/(y^2 - x^3)$.

Geometry and Parametrization: This defines a plane curve with a "cusp" at the origin. We can define a ring homomorphism $\phi : A \to k[t]$ by setting $x \mapsto t^2$ and $y \mapsto t^3$. This is injective, identifying A with the subring $k[t^2, t^3]$ of $k[t]$. The corresponding morphism of schemes $f : \mathbb{A}^1_k \to X$ is a homeomorphism on topological spaces (it is a bijection on points).

The Cuspidal Cubic $y^2 = x^3$



Singularity and Non-Isomorphism: Although f is a homeomorphism, it is **not** an isomorphism of schemes.

- The ring $A \cong k[t^2, t^3]$ is not integrally closed (the element $t = y/x$ is in the fraction field and satisfies $T^2 - x = 0$, but is not in A). Thus X is not normal.
- Examining the cotangent space at the origin $O = (x, y)$: The maximal ideal is $\mathfrak{m} = (x, y)$. The square of the maximal ideal contains x^2, xy, y^2. Note that the defining equation $y^2 - x^3 = 0$ involves terms of degree ≥ 2. In the quotient $\mathfrak{m}/\mathfrak{m}^2$, the relation becomes $0 = 0$. Thus, the Zariski cotangent space $\mathfrak{m}/\mathfrak{m}^2$ has basis $\{\bar{x}, \bar{y}\}$ and dimension 2.

Since $\dim T_O X = 2$ but $\dim X = 1$, the point O is a singular point. The "normalization" of X is the affine line Spec$k[t]$, which "smooths out" the cusp.

Example 46 The Arithmetic Surface: Spec$\mathbb{Z}[x]$.

Let $X = \operatorname{Spec}\mathbb{Z}[x]$. This scheme is the fundamental object linking number theory and geometry, often visualized as a surface fibered over the "arithmetic curve" Spec$\mathbb{Z}$.

Krull Dimension and Point Classification. The ring $\mathbb{Z}[x]$ has Krull dimension 2. Its prime ideals (points of X) are classified by their height (codimension):

(I) **Closed Points (Height 2):** These are maximal ideals of the form $\mathfrak{m} = (p, f(x))$, where $p \in \mathbb{Z}$ is a prime number and $f(x) \in \mathbb{Z}[x]$ is a monic polynomial that is irreducible modulo p. The residue field is a finite field:

$$\kappa(\mathfrak{m}) = \mathbb{Z}[x]/(p, f(x)) \cong \mathbb{F}_p[x]/(\bar{f}(x)) \cong \mathbb{F}_{p^d},$$

where $d = \deg(f)$. Geometrically, these points appear in the "fiber" over p.

(II) **Curve-like Points (Height 1):** These are non-maximal prime ideals, representing irreducible curves on the surface. They fall into two types:

- *"Vertical" Fibers:* Ideals of the form $\mathfrak{p} = (p)$. The closure $V(p)$ is isomorphic to the affine line over $\mathbb{F}_p$, i.e., $\mathrm{Spec}\mathbb{F}_p[x]$. The projection $\pi : X \to \mathrm{Spec}\mathbb{Z}$ maps this entire line to the closed point $(p) \in \mathrm{Spec}\mathbb{Z}$.
- *"Horizontal" Curves:* Ideals of the form $\mathfrak{p} = (f(x))$, where $f(x)$ is irreducible in $\mathbb{Z}[x]$ (and primitive). The closure $V(f)$ maps dominantly to $\mathrm{Spec}\mathbb{Z}$. This corresponds to an algebraic number defined by $f(\alpha) = 0$, spread out over the integers.

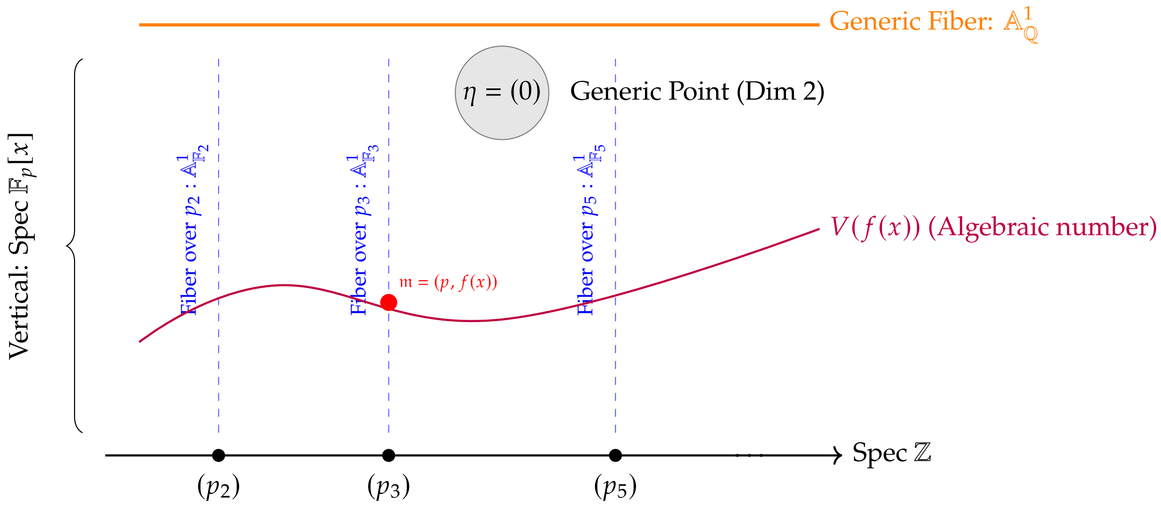
(III) **The Generic Point (Height 0):** The zero ideal (0). Its closure is the entire space X.

Topology and Closed Sets. The closed sets of X are finite unions of:

- The whole space X (dimension 2).
- "Vertical" lines (fibers over primes p).
- "Horizontal" curves (defined by equations $f(x) = 0$).
- Individual closed points (intersections of vertical and horizontal curves).

This topology captures the arithmetic intuition: finding integer solutions to $f(x) \equiv 0 \pmod{p}$ corresponds geometrically to finding the intersection points of the horizontal curve $V(f)$ with the vertical fiber $V(p)$.

The Arithmetic Surface Spec $\mathbb{Z}[x]$



Visualization (Mumford's Picture). One typically visualizes X as a plane where the horizontal axis represents prime numbers $2, 3, 5, \ldots$ and the vertical axis represents the variable x. The fiber over (0) is the "generic fiber" $\mathrm{Spec}\mathbb{Q}[x]$ (affine line over $\mathbb{Q}$), while fibers over (p) are lines with "finite geometry" characteristic p.

Example 47 Affine Line over a Finite Field: $\mathrm{Spec}\mathbb{F}_q[x]$.

Let $k = \mathbb{F}_q$ be a finite field with q elements, and let $X = \mathbb{A}^1_k = \mathrm{Spec}k[x]$.

Topological Structure. Since $k[x]$ is a Principal Ideal Domain (PID), the topological structure is ostensibly similar to $\mathrm{Spec}\mathbb{C}[x]$ or $\mathrm{Spec}\mathbb{Z}$: it carries the cofinite topology.

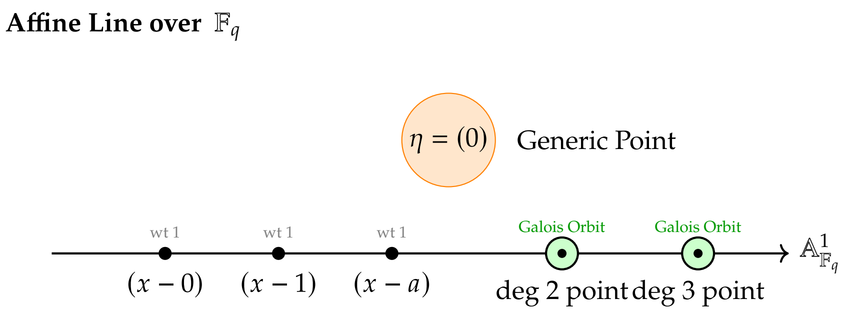
- The generic point corresponds to (0).
- The closed points correspond to maximal ideals $\mathfrak{m} = (f(x))$, where f is a monic irreducible polynomial in $k[x]$.

Arithmetic of Closed Points. Unlike the algebraically closed case ($\mathrm{Spec}\mathbb{C}[x]$), the closed points here have "internal structure" and varying "weights."

- If $f(x) = x - a$ (degree 1), the point corresponds to an element $a \in k$. The residue field is k. There are exactly q such points ("rational points").
- If $\deg(f) = d > 1$, the ideal (f) represents a *single* point in the scheme X, but its residue field is a finite extension:

  $$\kappa((f)) = k[x]/(f) \cong \mathbb{F}_{q^d}.$$

  Geometrically, this single closed point corresponds to a **Galois orbit** of d distinct geometric points in the algebraic closure $\mathrm{Spec}\bar{k}[x]$.



Key Features:
1. **Points**: Closed points are irreducible polynomials $f(x) \in \mathbb{F}_q[x]$.
2. **Weights**: Degree d points have residue field $\mathbb{F}_{q^d}$.
3. **Geometry**: A degree d point is a Galois orbit of d points in $\mathbb{A}^1_{\bar{\mathbb{F}}_q}$.

Zeta Function Interpretation. The topology of X is determined by the distribution of irreducible polynomials. The number of closed points is infinite, but for any fixed degree d, there are finitely many. This counting data is encoded in the Hasse-Weil Zeta function of the scheme:

$$Z(X,t) = \exp\left(\sum_{n=1}^{\infty} \frac{|X(\mathbb{F}_{q^n})|}{n} t^n\right) = \frac{1}{1-qt}.$$

This reflects that topologically (in the étale sense), $\mathrm{Spec}\mathbb{F}_q[x]$ behaves like a simply connected curve with a fundamental group generated by the Frobenius automorphism.

Example 48 Spectra of Fields and Base Change.

Consider the spectra of the fundamental fields of characteristic zero: $X_{\mathbb{Q}} = \mathrm{Spec}\mathbb{Q}$, $X_{\mathbb{R}} = \mathrm{Spec}\mathbb{R}$, and $X_{\mathbb{C}} = \mathrm{Spec}\mathbb{C}$.

1. The Structure of a Single Point. Topologically, the spectrum of any field k is a single point $\{(0)\}$, as the only prime ideal in a field is the zero ideal. However, in the category of schemes, these points are distinguished by their **residue fields**.

- A morphism $\mathrm{Spec}L \to \mathrm{Spec}K$ exists if and only if there is a field embedding $K \hookrightarrow L$.
- Thus, we have a sequence of canonical morphisms (morphisms of S-schemes over $\mathrm{Spec}\mathbb{Q}$):
$$\mathrm{Spec}\mathbb{C} \to \mathrm{Spec}\mathbb{R} \to \mathrm{Spec}\mathbb{Q}.$$

2. Base Change via Fiber Product. The power of the fiber product lies in its ability to "extend scalars" geometrically. Let X be a scheme over S. For any morphism $S' \to S$, the **base change** of X to S' is defined as $X_{S'} = X \times_S S'$.

In the affine case, if $X = \mathrm{Spec}B$ is an A-scheme (where $S = \mathrm{Spec}A$) and we have a ring map $A \to A'$ (where $S' = \mathrm{Spec}A'$), then:

$$X_{S'} = \mathrm{Spec}(B \otimes_A A').$$

3. Concrete Application: Geometric Realization of Galois Theory. Consider the scheme $X = \mathrm{Spec}(\mathbb{R}[x]/(x^2+1))$. Topologically, this is a single point because (x^2+1) is irreducible over $\mathbb{R}$ (it is a closed point with residue field $\mathbb{C}$).

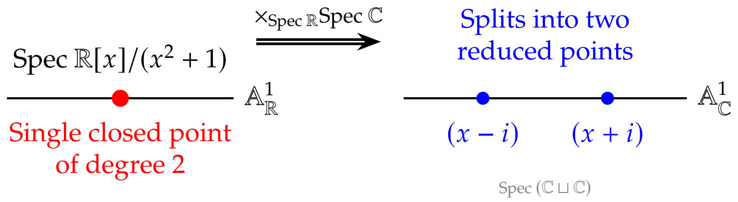
Now, let us perform a **base change** from $\mathbb{R}$ to $\mathbb{C}$. We compute the fiber product $X_{\mathbb{C}} = X \times_{\mathrm{Spec}\mathbb{R}} \mathrm{Spec}\mathbb{C}$:

$$\begin{aligned} X \times_{\mathrm{Spec}\mathbb{R}} \mathrm{Spec}\mathbb{C} &\cong \mathrm{Spec}\left(\frac{\mathbb{R}[x]}{(x^2+1)} \otimes_{\mathbb{R}} \mathbb{C}\right) \\ &\cong \mathrm{Spec}\left(\mathbb{C}[x]/(x^2+1)\right) \\ &\cong \mathrm{Spec}\left(\mathbb{C}[x]/(x-i) \times \mathbb{C}[x]/(x+i)\right) \\ &\cong \mathrm{Spec}\mathbb{C} \sqcup \mathrm{Spec}\mathbb{C}. \end{aligned}$$

Category of Points (distinguished by residue fields)

$$\begin{tikzcd}
\operatorname{Spec}\mathbb{C} \arrow[r, "\pi_{\mathbb{C}/\mathbb{R}}"] & \operatorname{Spec}\mathbb{R} \arrow[r, "\pi_{\mathbb{R}/\mathbb{Q}}"] & \operatorname{Spec}\mathbb{Q}
\end{tikzcd}$$

Concrete Application: Base Change of $V(x^2+1)$



Geometric Interpretation:
• **Over** $\mathbb{R}$: The point (x^2+1) is a single "fat" point of degree 2 (no rational coordinates).
• **Base Change**: Moving to the algebraic closure $\mathbb{C}$ "resolves" the equation.
• **Splitting**: The fiber product $\operatorname{Spec}(\mathbb{R}[x]/(x^2+1) \otimes_{\mathbb{R}} \mathbb{C})$ results in a disjoint union of two points.

Geometric Interpretation:

- Over $\mathbb{R}$, the "variety" $x^2+1=0$ has no rational points, but it exists as a "closed point of degree 2."
- After base change to $\mathbb{C}$ (the algebraic closure), this single point **splits** into two distinct reduced points, $\{i,-i\}$.
- This illustrates that the fiber product is the geometric mechanism for studying how algebraic equations behave when the field of coefficients is enlarged.

4. Base Change of the Affine Line. Consider the affine line over $\mathbb{Q}$, $X = \mathbb{A}^1_{\mathbb{Q}} = \operatorname{Spec}\mathbb{Q}[t]$.

- The closed points of $\mathbb{A}^1_{\mathbb{Q}}$ are orbits of the absolute Galois group $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For instance, the point corresponding to (t^2-2) is a single closed point.
- Upon base change to $\mathbb{R}$, $X_{\mathbb{R}} \cong \mathbb{A}^1_{\mathbb{R}}$. The point (t^2-2) splits into two points $(t-\sqrt{2})$ and $(t+\sqrt{2})$.
- Upon base change to $\mathbb{Q}(i)$, the ideal (t^2-2) remains prime (irreducible), so the point does not split.

Thus, $\times_S S'$ allows us to view the same "geometric shape" (the affine line) over different arithmetic contexts.

CHAPTER 4

Cohomology of Sheaves on Affine Schemes

4.1 Motives and Basic Setup of Cohomology of Sheaves

Intuition 1 .

In geometry, we often encounter the following problem: we have a short exact sequence of sheaves

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0.$$

Exactness means that locally (at the level of stalks), the map $\mathscr{F}_x \to \mathscr{F}''_x$ is surjective. In other words, for any section $s'' \in \mathscr{F}''(U)$, we can find a small neighborhood $V \subset U$ and a section $s \in \mathscr{F}(V)$ mapping to $s''|_V$.

The Global Question: Can we find a *global* section $s \in \mathscr{F}(X)$ that maps to a given global section $s'' \in \mathscr{F}''(X)$?

Generally, the answer is **no**. We can solve the problem locally, but these local solutions might not glue together to form a global solution. The **first cohomology group** $H^1(X, \mathscr{F}')$ precisely measures the obstruction to this gluing process. This leads to the long exact sequence:

$$0 \to \mathscr{F}'(X) \to \mathscr{F}(X) \to \mathscr{F}''(X) \xrightarrow{\delta} H^1(X, \mathscr{F}') \to \dots$$

Proposition 2 Left Exactness of the Global Section Functor.

Let X be a topological space. Let

$$0 \longrightarrow \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''$$

be an exact sequence of sheaves of abelian groups on X. Then the induced sequence of abelian groups of global sections is exact:

$$0 \longrightarrow \Gamma(X, \mathscr{F}') \xrightarrow{\phi_X} \Gamma(X, \mathscr{F}) \xrightarrow{\psi_X} \Gamma(X, \mathscr{F}'').$$

Furthermore, this result holds for any open subset $U \subseteq X$, meaning the functor $\Gamma(U, -)$ is left exact.

Proof. The exactness of the sheaf sequence means that for every point $x \in X$, the sequence of stalks $0 \longrightarrow \mathscr{F}'_x \xrightarrow{\phi_x} \mathscr{F}_x \xrightarrow{\psi_x} \mathscr{F}''_x$ is exact.

Step 1: Injectivity of ϕ_X. Let $s \in \Gamma(X, \mathscr{F}')$ such that $\phi_X(s) = 0$. This means the section $\phi_X(s)$ is the zero section. For any point $x \in X$, the germ satisfies $(\phi_X(s))_x = \phi_x(s_x) = 0$. Since ϕ_x is injective (by stalk exactness), we have $s_x = 0$ for all x. Since a section is determined by its germs, $s = 0$. Thus ϕ_X is injective.

Step 2: Exactness at $\Gamma(X, \mathscr{F})$. We need to show $\mathrm{Im}(\phi_X) = \mathrm{Ker}(\psi_X)$.

($\subseteq$) Since $\psi \circ \phi = 0$ as a sheaf morphism, the composition of section maps $\psi_X \circ \phi_X$ is also the zero map. Thus $\mathrm{Im}(\phi_X) \subseteq \mathrm{Ker}(\psi_X)$.

($\supseteq$) Let $s \in \Gamma(X, \mathscr{F})$ such that $\psi_X(s) = 0$. We must construct a section $t \in \Gamma(X, \mathscr{F}')$ such that $\phi_X(t) = s$. Consider a point $x \in X$. The germ $s_x \in \mathscr{F}_x$ satisfies $\psi_x(s_x) = (\psi_X(s))_x = 0$. By the exactness at the stalk level, $\mathrm{Ker}(\psi_x) = \mathrm{Im}(\phi_x)$. Therefore, there exists a unique germ $t_x \in \mathscr{F}'_x$ such that $\phi_x(t_x) = s_x$ (uniqueness follows from the injectivity of ϕ_x).

We now need to show that these germs t_x glue together to form a continuous global section. Since t_x comes from a stalk, there exists an open neighborhood U_x of x and a local section $\tau^{(x)} \in \mathscr{F}'(U_x)$ such that $(\tau^{(x)})_x = t_x$. Applying ϕ, the section $\phi(\tau^{(x)}) \in \mathscr{F}(U_x)$ has the same germ at x as $s|_{U_x}$. Thus, by shrinking U_x if necessary, we can assume $\phi(\tau^{(x)}) = s|_{U_x}$ on the entire neighborhood U_x.

On the overlap $U_x \cap U_y$, we have two sections $\tau^{(x)}$ and $\tau^{(y)}$ both mapping to s via ϕ. Since ϕ is injective (as proved in Step 1 for any open set), $\tau^{(x)}$ and $\tau^{(y)}$ must coincide on $U_x \cap U_y$. By the sheaf axiom (Gluing), there exists a unique global section $t \in \Gamma(X, \mathscr{F}')$ such that $t|_{U_x} = \tau^{(x)}$. By construction, $\phi_X(t) = s$. □

Remark: Generality of the Statement

The proof above relies *only* on the definition of a sheaf and the exactness of stalks. It does not use the specific structure of affine schemes. Therefore, this proposition holds for:

- Sheaves on arbitrary topological spaces.
- Sheaves of smooth functions on differentiable manifolds.
- Sheaves of holomorphic functions on complex manifolds.

This universality makes sheaf cohomology a fundamental tool across all geometry.

Example 3 Failure of Right Exactness: The Exponential Sequence.

Let $X = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ be the punctured complex plane (viewed as a complex manifold). Consider the **exponential sequence** of sheaves:

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\cdot 2\pi i} \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^* \longrightarrow 1.$$

Here:

- $\underline{\mathbb{Z}}$ is the **constant sheaf** of integers.

- $\mathscr{O}_X$ is the sheaf of holomorphic functions (additive group).
- $\mathscr{O}_X^*$ is the sheaf of non-vanishing holomorphic functions (multiplicative group).
- The map exp is given by $f \mapsto e^f$.

Local Exactness: Locally, for any small disk $U \subset \mathbb{C}^*$, any non-vanishing holomorphic function $g \in \mathscr{O}_X^*(U)$ has a logarithm $\log g$. Thus the map $\exp : \mathscr{O}_X(U) \to \mathscr{O}_X^*(U)$ is surjective. The sequence is exact as sheaves.

Global Failure: Consider the coordinate function $z \in \Gamma(X, \mathscr{O}_X^*)$. This is a globally defined, non-vanishing holomorphic function. Does there exist a global section $f \in \Gamma(X, \mathscr{O}_X)$ such that $e^f = z$? Such an f would be a global branch of the logarithm, $f = \log z$. However, as we traverse a loop around the origin, the argument of z increases by 2π. The function $\log z$ is multi-valued and cannot be defined as a continuous single-valued function on the whole of $\mathbb{C}^*$.

Therefore, z is not in the image of the global map $\Gamma(X, \mathscr{O}_X) \xrightarrow{\exp} \Gamma(X, \mathscr{O}_X^*)$. The sequence of global sections is **not right exact**.

This failure is captured by the first cohomology group $H^1(\mathbb{C}^*, \underline{\mathbb{Z}}) \cong \mathbb{Z}$, which corresponds to the winding number obstruction.

Proposition 4 Enough Injectives in $\mathsf{Mod}(X)$.

Let $(X, \mathscr{O}_X)$ be a ringed space. The category $\mathsf{Mod}(X)$ of sheaves of $\mathscr{O}_X$-modules has **enough injectives**. That is, for any sheaf $\mathscr{F}$, there exists an injective sheaf $\mathscr{I}$ and a monomorphism $\mathscr{F} \hookrightarrow \mathscr{I}$.

Proof. The construction relies on the existence of enough injectives in the category of modules over a ring. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. For each point $x \in X$, the stalk $\mathscr{F}_x$ is a module over the local ring $\mathscr{O}_{X,x}$. Since the category of $\mathscr{O}_{X,x}$-modules possesses enough injectives, there exists an embedding $\mathscr{F}_x \hookrightarrow I_x$, where I_x is an injective $\mathscr{O}_{X,x}$-module.

We construct the sheaf $\mathscr{I}$ as the product of "skyscraper" sheaves supported at each point. Let $j_x : \{x\} \hookrightarrow X$ denote the inclusion of the point x. We consider I_x as a sheaf on the singleton space $\{x\}$ and form the direct image sheaf $j_{x*}(I_x)$ on X. Explicitly, the sections of $j_{x*}(I_x)$ on an open set U are I_x if $x \in U$, and 0 otherwise. Define $\mathscr{I}$ to be the product of these sheaves over all points in X:

$$\mathscr{I} = \prod_{x \in X} j_{x*}(I_x).$$

There is a canonical morphism $\phi : \mathscr{F} \to \mathscr{I}$. To define this, recall that giving a morphism into a product is equivalent to giving morphisms into each factor. For each x, the map $\mathscr{F} \to j_{x*}(I_x)$ corresponds, via the adjunction between the inverse image functor j_x^* (taking the stalk) and the direct image functor j_{x*}, to a map of stalks $\mathscr{F}_x \to I_x$. We choose this to be the embedding fixed earlier. Since the map ϕ induces an injection on every stalk (by construction), it is a monomorphism of sheaves.

It remains to prove that the sheaf $\mathscr{I}$ is an injective object in $\mathsf{Mod}(X)$. A sheaf $\mathscr{I}$ is injective if the functor $\mathrm{Hom}_{\mathscr{O}_X}(-, \mathscr{I})$ is exact. For any sheaf $\mathscr{G}$, we utilize the properties of the product and the adjunction mentioned above:

$$\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{G}, \mathscr{I}) \cong \operatorname{Hom}_{\mathscr{O}_X}\left(\mathscr{G}, \prod_{x \in X} j_{x*}(I_x)\right) \cong \prod_{x \in X} \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{G}, j_{x*}(I_x)).$$

By the adjunction $j_x^* \dashv j_{x*}$, we have $\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{G}, j_{x*}(I_x)) \cong \operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{G}_x, I_x)$. Thus, the functor in question is isomorphic to the composition of the stalk functor $\mathscr{G} \mapsto \{\mathscr{G}_x\}_x$ and the product of Hom functors $\prod \operatorname{Hom}_{\mathscr{O}_{X,x}}(-, I_x)$. Since the stalk functor is exact, and each I_x is an injective module (making $\operatorname{Hom}_{\mathscr{O}_{X,x}}(-, I_x)$ exact), and the direct product of exact sequences of abelian groups is exact, the total functor $\operatorname{Hom}_{\mathscr{O}_X}(-, \mathscr{I})$ is exact. Consequently, $\mathscr{I}$ is an injective sheaf. □

Corollary 5 Enough Injectives for Abelian Sheaves.

The category of sheaves of abelian groups on any topological space X, denoted by $\mathfrak{Ab}(X)$, has enough injectives.

Proof. A sheaf of abelian groups is precisely a sheaf of modules over the constant sheaf of rings $\underline{\mathbb{Z}}_X$. By applying **Proposition** 4 to the ringed space $(X, \underline{\mathbb{Z}}_X)$, the result follows immediately. □

Definition 6 Sheaf Cohomology.

Let $(X, \mathscr{O}_X)$ be a ringed space. We denote by $\mathsf{Mod}(X)$ the category of sheaves of $\mathscr{O}_X$-modules. The **sheaf cohomology functors**, denoted by $H^i(X, \cdot)$, are defined as the right derived functors of the global section functor $\Gamma(X, \cdot)$.

Specifically, for any sheaf $\mathscr{F} \in \mathsf{Mod}(X)$, the cohomology groups $H^i(X, \mathscr{F})$ are computed by choosing an injective resolution of $\mathscr{F}$:

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^0 \longrightarrow \mathscr{I}^1 \longrightarrow \mathscr{I}^2 \longrightarrow \dots$$

where each $\mathscr{I}^k$ is an injective object in $\mathsf{Mod}(X)$. Applying the global section functor yields a complex:

$$0 \longrightarrow \Gamma(X, \mathscr{I}^0) \longrightarrow \Gamma(X, \mathscr{I}^1) \longrightarrow \Gamma(X, \mathscr{I}^2) \longrightarrow \dots$$

The cohomology groups are defined as the cohomology of this complex:

$$H^i(X, \mathscr{F}) = h^i(\Gamma(X, \mathscr{I}^\bullet)) = \frac{\operatorname{Ker}(\Gamma(X, \mathscr{I}^i) \to \Gamma(X, \mathscr{I}^{i+1}))}{\operatorname{Im}(\Gamma(X, \mathscr{I}^{i-1}) \to \Gamma(X, \mathscr{I}^i))}.$$

Remark: Although this chapter focuses primarily on affine schemes where $\mathscr{O}_X$-modules are quasi-coherent, this definition is valid for any ringed space, including general schemes, complex analytic spaces, and differentiable manifolds. The existence of these groups is guaranteed by the following proposition.

Definition 7 Flasque Sheaves.

A sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is said to be **flasque** (or *flabby*) if for every inclusion of open subsets $V \subseteq U \subseteq X$, the restriction map

$$\rho_{UV} : \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$$

is surjective. In particular, this implies that the map from global sections $\mathscr{F}(X) \to \mathscr{F}(U)$ is surjective for any open set U.

Proposition 8 Injectives are Flasque.

Let $(X, \mathscr{O}_X)$ be a ringed space. Every injective $\mathscr{O}_X$-module $\mathscr{I}$ is flasque.

Proof. To prove the surjectivity of the restriction map $\mathscr{I}(U) \to \mathscr{I}(V)$, we translate this problem into the exactness of the Hom functor by constructing specific test sheaves.

For any open set $W \subseteq X$, let $j_W : W \hookrightarrow X$ be the inclusion map. We define the **extension by zero** sheaf, denoted by $\mathcal{Z}_W$ (standardly $j_{W!}(\mathscr{O}_X|_W)$), as the sheaf on X associated to the presheaf:

$$\mathcal{Z}_W(K) = \begin{cases} \mathscr{O}_X(K) & \text{if } K \subseteq W \\ 0 & \text{otherwise} \end{cases}$$

More precisely, at the level of stalks, $(\mathcal{Z}_W)_x \cong \mathscr{O}_{X,x}$ if $x \in W$ and $(\mathcal{Z}_W)_x = 0$ if $x \notin W$.

A crucial property of this sheaf is the representability of sections over W. For any $\mathscr{O}_X$-module $\mathscr{F}$, there is a canonical isomorphism:

$$\operatorname{Hom}_{\mathscr{O}_X}(\mathcal{Z}_W, \mathscr{F}) \cong \mathscr{F}(W).$$

This isomorphism maps a morphism $\phi : \mathcal{Z}_W \to \mathscr{F}$ to the section $\phi_W(1_W) \in \mathscr{F}(W)$, where 1_W is the identity section of $\mathscr{O}_X$ restricted to W.

Now, consider two open sets $V \subseteq U$. Since the support of $\mathcal{Z}_V$ is contained in the support of $\mathcal{Z}_U$, and they behave identically to the structure sheaf inside their supports, there is a natural monomorphism of sheaves:

$$0 \longrightarrow \mathcal{Z}_V \longrightarrow \mathcal{Z}_U.$$

This map is the identity on stalks for $x \in V$, and the zero map $0 \to \mathscr{O}_{X,x}$ for $x \in U \setminus V$.

Since $\mathscr{I}$ is an injective object in the category of $\mathscr{O}_X$-modules, the functor $\operatorname{Hom}_{\mathscr{O}_X}(-, \mathscr{I})$ is exact (specifically, it maps monomorphisms to epimorphisms). Applying this functor to the inclusion above yields a surjective map:

$$\operatorname{Hom}_{\mathscr{O}_X}(\mathcal{Z}_U, \mathscr{I}) \longrightarrow \operatorname{Hom}_{\mathscr{O}_X}(\mathcal{Z}_V, \mathscr{I}) \longrightarrow 0.$$

Using the canonical identification $\operatorname{Hom}_{\mathscr{O}_X}(\mathcal{Z}_W, \mathscr{I}) \cong \mathscr{I}(W)$, this sequence is precisely:

$$\mathscr{I}(U) \xrightarrow{\rho_{UV}} \mathscr{I}(V) \longrightarrow 0.$$

Thus, the restriction map is surjective for any $V \subseteq U$, proving that $\mathscr{I}$ is flasque. □

Note 9 Independence of the Ambient Category.

The definition of sheaf cohomology via derived functors is robust. While we initially defined $H^i(X, \mathscr{F})$ using injective resolutions in the category $\mathsf{Mod}(X)$ of $\mathscr{O}_X$-modules, we could equivalently treat $\mathscr{F}$ as an object in the category $\mathfrak{Ab}(X)$ of sheaves of abelian groups.

The bridge between these viewpoints is the concept of **flasque sheaves**. As we shall see, injective objects in both categories are flasque. Moreover, flasque sheaves are acyclic for the global section functor. Consequently, a resolution by flasque $\mathscr{O}_X$-modules is simultaneously a flasque resolution of abelian sheaves. By the general theory of derived functors (specifically, the comparison theorem for resolutions by acyclic objects), both approaches yield isomorphic cohomology groups. Thus, we may freely use algebraic properties of modules to compute what is essentially a topological invariant.

Lemma 10 Properties of Flasque Sheaves.

Let X be a topological space.

(1) If X is irreducible, then any constant sheaf A_X (where A is an abelian group) is flasque.

(2) Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of sheaves. If $\mathscr{F}'$ is flasque, then the sequence of global sections

$$0 \longrightarrow \Gamma(X, \mathscr{F}') \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}'') \longrightarrow 0$$

is exact (i.e., the global section functor is exact on such sequences).

(3) In the situation of (2), if $\mathscr{F}'$ and $\mathscr{F}$ are flasque, then $\mathscr{F}''$ is also flasque.

(4) Let $f : X \to Y$ be a continuous map. If $\mathscr{F}$ is a flasque sheaf on X, then the direct image sheaf $f_*\mathscr{F}$ is a flasque sheaf on Y.

(5) The sheaf of discontinuous sections (the Godement sheaf) is flasque.

Proof. For (1), recall that on an irreducible space, every non-empty open set U is dense and connected. Consequently, the restriction map for a constant sheaf corresponds to the identity map $A \to A$ (or $A \to 0$ if the target is empty), which is trivially surjective.

For (2), left exactness is known. We must prove the surjectivity of $\Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'')$. Let $s'' \in \Gamma(X, \mathscr{F}'')$. Since the sheaf map is surjective, there exists an open cover $\{U_i\}$ and local sections $s_i \in \mathscr{F}(U_i)$ mapping to $s''|_{U_i}$. On the intersection $U_i \cap U_j$, the difference $s_i - s_j$ maps to zero in $\mathscr{F}''$, so it comes from a section $t_{ij} \in \mathscr{F}'(U_i \cap U_j)$. The collection $\{t_{ij}\}$ forms a 1-cocycle. To glue the s_i into a global section, we essentially need to solve a "cohomological problem." However, since $\mathscr{F}'$ is flasque, we can argue by extension. Consider the set of pairs (U, s) where U is open and s lifts $s''|_U$, ordered by inclusion. By Zorn's Lemma, there exists a maximal element (U_{max}, s_{max}). If $U_{max} \neq X$, we can pick a point $x \notin U_{max}$ and a local lift s_x on a neighborhood V. On $U_{max} \cap V$, the difference is a section of $\mathscr{F}'$. Since $\mathscr{F}'$ is flasque, this difference extends to V. Using this extension, we can adjust s_x to match s_{max} on the overlap, extending the lift to $U_{max} \cup V$, contradicting maximality. Thus a global lift exists.

For (3), let $V \subseteq U$ be open sets. Consider the commutative diagram with exact rows (by part (2) applied to the restriction to U and V):

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}'(U) \arrow[r] \arrow[d, "res'"] & \mathscr{F}(U) \arrow[r] \arrow[d, "res"] & \mathscr{F}''(U) \arrow[r] \arrow[d, "res''"] & 0 \\
0 \arrow[r] & \mathscr{F}'(V) \arrow[r] & \mathscr{F}(V) \arrow[r] & \mathscr{F}''(V) \arrow[r] & 0
\end{tikzcd}$$

Since $\mathscr{F}'$ is flasque, res' is surjective. Since $\mathscr{F}$ is flasque, res is surjective. By the Snake Lemma (or a simple diagram chase), the map res'' must be surjective. Thus $\mathscr{F}''$ is flasque.

For (4), the result follows immediately from the definition: for $V \subseteq U \subseteq Y$, the map $(f_*\mathscr{F})(U) \to (f_*\mathscr{F})(V)$ is simply the map $\mathscr{F}(f^{-1}(U)) \to \mathscr{F}(f^{-1}(V))$. Since $f^{-1}(V) \subseteq f^{-1}(U)$ and $\mathscr{F}$ is flasque, this is surjective. □

Lemma 11 Direct Limits and Flasque Sheaves.

Let X be a Noetherian topological space.

(1) Let $\{\mathscr{F}_i\}_{i\in I}$ be a direct system of sheaves on X. Then the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$ is a sheaf. In other words, the direct limit functor commutes with the section functor on Noetherian spaces.

(2) If each $\mathscr{F}_i$ is flasque, then the direct limit sheaf $\varinjlim \mathscr{F}_i$ is also flasque.

Proof. For (1), we verify the sheaf axioms for the presheaf $P(U) = \varinjlim \mathscr{F}_i(U)$. The axiom involves an exact sequence of products for any open cover. On a Noetherian space, every open subset U is quasi-compact. This allows us to reduce any open cover of U to a **finite** subcover. The direct limit functor $\varinjlim$ commutes with finite limits (specifically finite products and kernels) in the category of abelian groups. Therefore, if we take the exact sequence defining the sheaf property for each $\mathscr{F}_i$ with respect to a finite cover, and apply the exact functor $\varinjlim$, we obtain the exact sequence for P. Since this holds for the finite subcover (which covers U), P satisfies the sheaf condition.

For (2), let $\mathscr{F} = \varinjlim \mathscr{F}_i$. We need to show that for $V \subseteq U$, the map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective. By part (1), we have identifications $\mathscr{F}(U) = \varinjlim(\mathscr{F}_i(U))$ and $\mathscr{F}(V) = \varinjlim(\mathscr{F}_i(V))$. Since each $\mathscr{F}_i$ is flasque, the restriction maps $\rho_i : \mathscr{F}_i(U) \to \mathscr{F}_i(V)$ are surjective. The direct limit of a system of surjective maps in the category of abelian groups is surjective. Thus, the induced map $\varinjlim \rho_i$ is surjective, proving that $\mathscr{F}$ is flasque. □

Proposition 12 Cohomology Commutes with Direct Limits.

Let X be a Noetherian topological space. Let $\{\mathscr{F}_\alpha\}_{\alpha\in I}$ be a direct system of sheaves of abelian groups on X. Then for every integer $i \geq 0$, there is a natural isomorphism:

$$\varinjlim H^i(X, \mathscr{F}_\alpha) \xrightarrow{\sim} H^i(X, \varinjlim \mathscr{F}_\alpha).$$

Proof. We proceed by comparing two cohomological δ-functors. Consider the category of direct systems of sheaves on X, denoted by C. The functor $\mathscr{F} \mapsto H^i(X, \mathscr{F})$ defines a δ-functor from C to the category of abelian groups. Similarly, the composite functor $\mathscr{F} \mapsto H^i(X, \varinjlim \mathscr{F})$ is also a δ-functor, since the direct limit functor $\varinjlim$ is exact on the category of abelian groups (and sheaves).

For the case $i = 0$, we established in **Lemma 11** that on a Noetherian space, the global section functor commutes with direct limits. Thus, we have a natural isomorphism $\varinjlim \Gamma(X, \mathscr{F}_\alpha) \cong \Gamma(X, \varinjlim \mathscr{F}_\alpha)$. To extend this isomorphism to higher cohomology, it suffices to show that both functors are *effaceable* for $i > 0$. By the general theory of universal δ-functors, if two δ-functors agree in degree 0 and are both effaceable in positive degrees, they are naturally isomorphic.

Let $\{\mathscr{F}_\alpha\}$ be an arbitrary direct system. For each α, we can embed $\mathscr{F}_\alpha$ into its Godement sheaf $\mathscr{G}_\alpha$ (the sheaf of discontinuous sections), which is flasque. This construction is functorial, so the sheaves $\{\mathscr{G}_\alpha\}$ form a direct system, and we have a monomorphism of direct systems $\{\mathscr{F}_\alpha\} \hookrightarrow \{\mathscr{G}_\alpha\}$. Since each $\mathscr{G}_\alpha$ is flasque, $H^i(X, \mathscr{G}_\alpha) = 0$ for all $i > 0$. Consequently, the direct limit of these groups, $\varinjlim H^i(X, \mathscr{G}_\alpha)$, vanishes for all $i > 0$. This shows that the functor $\varinjlim H^i(X, \cdot)$ is effaceable.

On the other hand, consider the limit sheaf $\mathscr{G} = \varinjlim \mathscr{G}_\alpha$. By **Lemma 11**, since X is Noetherian and each $\mathscr{G}_\alpha$ is flasque, the direct limit $\mathscr{G}$ is also flasque. Therefore, $H^i(X, \mathscr{G}) = 0$ for all $i > 0$, which implies that the functor $H^i(X, \varinjlim \cdot)$ is also effaceable. Since both functors are effaceable and coincide at $i = 0$, the natural homomorphism between them is an isomorphism for all i. □

Note 13 Noetherian Hypothesis.

The assumption that X is Noetherian is crucial here. It ensures that $\varinjlim$ produces a sheaf (not just a presheaf) and that it preserves the flasque property. Without this, the exactness required for the δ-functor argument would fail.

Lemma 14 Cohomology of Extension by Zero.

Let Y be a closed subset of a topological space X, and let $j : Y \hookrightarrow X$ be the inclusion map. Let $\mathscr{F}$ be a sheaf of abelian groups on Y. Let $j_*\mathscr{F}$ denote the direct image sheaf (which corresponds to the extension of $\mathscr{F}$ by zero outside Y). Then for all $i \geq 0$, there is a canonical isomorphism:

$$H^i(Y, \mathscr{F}) \cong H^i(X, j_*\mathscr{F}).$$

Proof. We compute the cohomology groups using flasque resolutions. Let $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ be a flasque resolution of $\mathscr{F}$ on the subspace Y. Since j is a continuous map, the direct image functor j_* is left exact. Moreover, since j is a closed immersion, the stalk of $j_*\mathscr{G}$ at a point x is $\mathscr{G}_x$ if $x \in Y$ and 0 if $x \notin Y$. This implies that j_* is actually an exact functor on the category of sheaves. Applying j_* to the resolution yields a complex $0 \to j_*\mathscr{F} \to j_*\mathscr{I}^\bullet$.

We previously established that the direct image of a flasque sheaf is flasque (**Lemma 10**). Therefore, $j_*\mathscr{I}^\bullet$ is a resolution of $j_*\mathscr{F}$ by flasque sheaves on X. To compute the cohomology $H^i(X, j_*\mathscr{F})$, we apply the global section functor $\Gamma(X, \cdot)$ to this resolution.

Observe that for any sheaf $\mathscr{G}$ on Y, the definition of the direct image gives $\Gamma(X, j_*\mathscr{G}) = \Gamma(j^{-1}(X), \mathscr{G}) = \Gamma(Y, \mathscr{G})$. Consequently, the complex of global sections $\Gamma(X, j_*\mathscr{I}^\bullet)$ is identical to the complex $\Gamma(Y, \mathscr{I}^\bullet)$. Since the cohomology groups are defined as the homology of these complexes, we conclude that $H^i(X, j_*\mathscr{F}) \cong H^i(Y, \mathscr{F})$. □

Theorem 15 Grothendieck Vanishing Theorem.

Let X be a Noetherian topological space of dimension n. Then for any sheaf of abelian groups $\mathscr{F}$ on X and for all $i > n$, we have:

$$H^i(X, \mathscr{F}) = 0.$$

Proof. We proceed by induction on the dimension n of X. The proof is divided into five rigorous steps, reducing the general case to successively simpler cases until we reach a direct calculation.

Step 1: Reduction to the case where X is irreducible. First, assume the theorem holds for all spaces of dimension $< n$. Now consider a space X of dimension n. Since X is Noetherian, it has finitely many irreducible components. We perform a secondary induction on the number of irreducible components of X.

If X is not irreducible, let Z be one of its irreducible components. Let $U = X \setminus Z$ be the complement. Note that Z is a closed subset and U is an open subset. We define two sheaves:

- $\mathscr{F}_Z = i_*(\mathscr{F}|_Z)$, the direct image of the restriction to Z (extension by zero outside Z).
- $\mathscr{F}_U = j_!(\mathscr{F}|_U)$, the extension by zero of the restriction to U.

There is a fundamental short exact sequence of sheaves on X:

$$0 \longrightarrow \mathscr{F}_U \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}_Z \longrightarrow 0.$$

Consider the long exact sequence of cohomology derived from this:

$$\cdots \longrightarrow H^i(X, \mathscr{F}_U) \longrightarrow H^i(X, \mathscr{F}) \longrightarrow H^i(X, \mathscr{F}_Z) \longrightarrow \dots$$

- For the term $H^i(X, \mathscr{F}_Z)$: By **Lemma 14**, $H^i(X, \mathscr{F}_Z) \cong H^i(Z, \mathscr{F}|_Z)$. Since Z is a closed subset, $\dim Z \leq n$. However, Z has strictly fewer irreducible components than X (unless X was already irreducible). By the secondary induction hypothesis, this group vanishes for $i > n$.
- For the term $H^i(X, \mathscr{F}_U)$: The sheaf $\mathscr{F}_U$ can be regarded as a sheaf on the closure $\bar{U}$. Since Z was a component, $\bar{U}$ has strictly fewer irreducible components than X. Again, by secondary induction, this vanishes.

Thus, $H^i(X, \mathscr{F}) = 0$. This allows us to assume henceforth that X is **irreducible** of dimension n.

Step 2: The base case $n = 0$. Suppose X is irreducible and $\dim X = 0$. Since X is Noetherian and irreducible, the closure of the generic point is the point itself. Thus X

consists of a single point (with non-Hausdorff geometric generic points absorbed). The category of sheaves on a single point is equivalent to the category of abelian groups $\mathfrak{Ab}$. The global section functor $\Gamma(X, \cdot)$ corresponds to the identity functor $\mathrm{Id} : \mathfrak{Ab} \to \mathfrak{Ab}$. Since the identity functor is exact, its derived functors vanish for $i > 0$. Thus $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.

Step 3: Reduction to sheaves generated by a single section. Let X be irreducible of dimension $n > 0$. Recall that any sheaf $\mathscr{F}$ is the direct limit of its finitely generated subsheaves. By **Proposition 12**, cohomology commutes with direct limits. Therefore, it suffices to prove the theorem for a sheaf $\mathscr{F}$ generated by a finite number of sections.

We proceed by induction on the number of generators. If $\mathscr{F}$ is generated by sections $s_1, \dots, s_k$, let $\mathscr{F}'$ be the subsheaf generated by $s_1, \dots, s_{k-1}$. We have an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, where $\mathscr{F}''$ is generated by a single section (the image of s_k). Using the long exact sequence, we reduce the problem to the case where $\mathscr{F}$ is generated by a single section s over some open set U.

Such a sheaf $\mathscr{F}$ is a quotient of the sheaf $\mathbb{Z}_U$ (the constant sheaf $\mathbb{Z}$ restricted to U and extended by zero to X). Let $\mathscr{R}$ be the kernel:

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathbb{Z}_U \longrightarrow \mathscr{F} \longrightarrow 0.$$

From the cohomology exact sequence, to prove $H^i(X, \mathscr{F}) = 0$ for $i > n$, it suffices to prove $H^i(X, \mathbb{Z}_U) = 0$ and $H^{i+1}(X, \mathscr{R}) = 0$. Since $i + 1 > n + 1 > n$, if we can prove the vanishing for *any* subsheaf of a constant sheaf and for the constant sheaf itself, we are done.

Step 4: Reduction for subsheaves of constant sheaves. Let $U \subseteq X$ be open and let $\mathscr{R} \subset \mathbb{Z}_U$ be a subsheaf. We want to show $H^i(X, \mathscr{R}) = 0$ for $i > n$. For any point $x \in U$, the stalk $\mathscr{R}_x$ is a subgroup of $(\mathbb{Z}_U)_x \cong \mathbb{Z}$. Thus $\mathscr{R}_x = d_x\mathbb{Z}$ for some integer $d_x \geq 0$. If $\mathscr{R} = 0$, the result is trivial. Otherwise, let $d = \min\{d_x \mid d_x > 0, x \in U\}$.

Since the generating section is locally constant, there exists a non-empty open subset $V \subseteq U$ such that for all $x \in V$, $\mathscr{R}_x = d\mathbb{Z}$. On this open set V, the sheaf $\mathscr{R}|_V$ is isomorphic to the constant sheaf $\mathbb{Z}|_V$ (via the isomorphism extending multiplication by d). Let $\mathbb{Z}_V$ denote the sheaf $j_{V!}(\mathbb{Z}|_V)$ on X. We have an inclusion $\mathbb{Z}_V \hookrightarrow \mathscr{R}$ (identifying $\mathbb{Z}_V$ with the subsheaf generated by sections of value d on V). Consider the sequence:

$$0 \longrightarrow \mathbb{Z}_V \longrightarrow \mathscr{R} \longrightarrow \mathscr{R}/\mathbb{Z}_V \longrightarrow 0.$$

The quotient sheaf $Q = \mathscr{R}/\mathbb{Z}_V$ has support in $U \setminus V$. Since X is irreducible, the complement of the non-empty open set V is a proper closed subset, which has dimension strictly less than n. By the induction hypothesis on dimension, $H^i(X, Q) = 0$ for $i > n$ (and even for $i \geq n$, actually, but $i > n$ suffices). Using the exact sequence, the vanishing of $H^i(X, \mathscr{R})$ is reduced to the vanishing of $H^i(X, \mathbb{Z}_V)$.

Step 5: Vanishing for extension of constant sheaves. It remains only to prove that for any open set $U \subseteq X$, $H^i(X, \mathbb{Z}_U) = 0$ for $i > n$. Let $Y = X \setminus U$ be the closed complement. We have the standard short exact sequence relating the extension by zero and the constant sheaf:

$$0 \longrightarrow \mathbb{Z}_U \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathbb{Z}_Y \longrightarrow 0,$$

where $\underline{\mathbb{Z}}$ is the constant sheaf on the whole space X, and $\mathbb{Z}_Y = i_*(\mathbb{Z}|_Y)$.

- The sheaf $\underline{\mathbb{Z}}$ is a constant sheaf on an **irreducible** space X. By **Lemma 10**, such a sheaf is flasque. Therefore, $H^i(X, \underline{\mathbb{Z}}) = 0$ for all $i > 0$.
- The sheaf $\mathbb{Z}_Y$ is supported on Y. Since X is irreducible and U is non-empty (otherwise $\mathbb{Z}_U = 0$ is trivial), $\dim Y < \dim X = n$. By the primary induction hypothesis on dimension, $H^{i-1}(X, \mathbb{Z}_Y) = 0$ for $i - 1 > n - 1$, which holds if $i > n$.

We examine the segment of the long exact sequence:

$$\cdots \longrightarrow H^{i-1}(X, \mathbb{Z}_Y) \longrightarrow H^i(X, \mathbb{Z}_U) \longrightarrow H^i(X, \underline{\mathbb{Z}}) \longrightarrow \ldots$$

For $i > n$, both the left term and the right term are zero. By exactness, $H^i(X, \mathbb{Z}_U) = 0$.

This completes the proof for the constant sheaf, which propagates back up through the reductions: from $\mathbb{Z}_U$ to $\mathscr{R}$, from $\mathscr{R}$ and $\mathbb{Z}_U$ to single-generator sheaves, to finitely generated sheaves, to arbitrary sheaves on irreducible spaces, and finally to arbitrary sheaves on Noetherian spaces. □

4.2 Čech Cohomology

Definition 16 Čech Complex.

Let X be a topological space and let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of X. We assume the index set I is well-ordered. Let $\mathscr{F}$ be a sheaf of abelian groups on X.

The **Čech complex** $C^\bullet(\mathfrak{U}, \mathscr{F})$ is defined as follows:

- For each integer $p \geq 0$, the group of p-cochains is the product of sections over all $(p+1)$-fold intersections:

$$C^p(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < i_1 < \cdots < i_p} \mathscr{F}(U_{i_0 i_1 \ldots i_p}),$$

  where $U_{i_0 \ldots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. An element $\alpha \in C^p(\mathfrak{U}, \mathscr{F})$ is determined by its components $\alpha_{i_0 \ldots i_p} \in \mathscr{F}(U_{i_0 \ldots i_p})$.

- The **coboundary map** $d^p : C^p(\mathfrak{U}, \mathscr{F}) \to C^{p+1}(\mathfrak{U}, \mathscr{F})$ is defined by the alternating sum of restrictions. For a cochain α, the component of $d\alpha$ at indices $i_0 < \cdots < i_{p+1}$ is given by:

$$(d\alpha)_{i_0 \ldots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \ldots \hat{i}_k \ldots i_{p+1}} |_{U_{i_0 \cdots i_{p+1}}},$$

  where $\hat{i}_k$ denotes the omission of the index i_k.

One can verify that $d^{p+1} \circ d^p = 0$, making $(C^\bullet(\mathfrak{U}, \mathscr{F}), d)$ a complex of abelian groups.

Definition 17 Čech Cohomology.

(1) **With respect to a covering:** The p-th Čech cohomology group of $\mathscr{F}$ with respect to the covering $\mathfrak{U}$ is the p-th cohomology of the Čech complex:

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) = h^p(C^\bullet(\mathfrak{U}, \mathscr{F})) = \frac{\operatorname{Ker}(d^p)}{\operatorname{Im}(d^{p-1})}.$$

(2) **Of the space** X**:** The Čech cohomology of the space X is defined as the direct limit over the directed set of all open coverings (ordered by refinement):

$$\check{H}^p(X, \mathscr{F}) = \varinjlim_{\mathfrak{U}} \check{H}^p(\mathfrak{U}, \mathscr{F}).$$

Note 18 Čech Cohomology is NOT a δ-Functor.

Unlike the derived functor cohomology $H^i(X, \cdot)$, the Čech cohomology does not, in general, yield a long exact sequence from a short exact sequence of sheaves.

Consider a short exact sequence of sheaves:

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0.$$

While this sequence is exact on stalks, it does **not** imply that the sequence of complexes

$$0 \longrightarrow C^\bullet(\mathfrak{U}, \mathscr{F}') \longrightarrow C^\bullet(\mathfrak{U}, \mathscr{F}) \longrightarrow C^\bullet(\mathfrak{U}, \mathscr{F}'') \longrightarrow 0$$

is exact. specifically, the map $C^p(\mathfrak{U}, \mathscr{F}) \to C^p(\mathfrak{U}, \mathscr{F}'')$ fails to be surjective. A section $s'' \in \mathscr{F}''(U_{i_0 \dots i_p})$ might not lift to a section in $\mathscr{F}$ over the same intersection $U_{i_0 \dots i_p}$. It only lifts locally.

Example of Failure: Take the trivial covering $\mathfrak{U} = \{X\}$. The Čech complex is simply the global sections:

$$0 \to \mathscr{F}'(X) \to \mathscr{F}(X) \to \mathscr{F}''(X) \to 0.$$

As we saw in **Example** 3 (Exponential Sequence), the map $\Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'')$ is not surjective. Thus, we cannot define a connecting homomorphism to link $\check{H}^0$ to $\check{H}^1$ in a long exact sequence. This defect is why derived functor cohomology is the theoretically superior definition, although Čech cohomology agrees with it on good spaces (like Noetherian separated schemes).

Lemma 19 Relation to Global Sections.

Let X be a topological space and $\mathfrak{U} = (U_i)_{i \in I}$ an open covering. For any sheaf of abelian groups $\mathscr{F}$, there is a canonical isomorphism:

$$\check{H}^0(\mathfrak{U}, \mathscr{F}) \cong \Gamma(X, \mathscr{F}).$$

Proof. Recall the definition of the 0-th cohomology group:

$$\check{H}^0(\mathfrak{U}, \mathscr{F}) = \operatorname{Ker}(d^0 : C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F})).$$

An element in $C^0(\mathfrak{U}, \mathscr{F})$ is a collection of local sections $s = (s_i)_{i \in I}$, where $s_i \in \mathscr{F}(U_i)$. The coboundary map d^0 is defined by:

$$(d^0 s)_{ij} = s_j|_{U_i \cap U_j} - s_i|_{U_i \cap U_j}.$$

Therefore, s belongs to the kernel if and only if $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all indices i, j.

This is precisely the **Gluing Axiom** for sheaves. A collection of compatible local sections $\{s_i\}$ glues uniquely to a global section $S \in \Gamma(X, \mathscr{F})$ such that $S|_{U_i} = s_i$. Conversely, any global section S defines a compatible collection via restriction. Thus, the kernel is isomorphic to the group of global sections. □

Definition 20 Čech Sheaf Complex.

Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of X. We define a complex of sheaves, denoted by $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$, called the **Čech sheaf complex**.

For each $p \geq 0$, the sheaf $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is defined as the product of the direct image sheaves from the intersections:

$$\mathscr{C}^p(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_p} (j_{i_0 \ldots i_p})_* \left(\mathscr{F}|_{U_{i_0 \ldots i_p}} \right),$$

where $j_{i_0 \ldots i_p} : U_{i_0 \ldots i_p} \hookrightarrow X$ is the inclusion map.

The boundary morphism $d : \mathscr{C}^p \to \mathscr{C}^{p+1}$ is defined locally using the same alternating sum formula as the standard Čech complex. A crucial property of this construction is that the group of global sections recovers the standard Čech complex:

$$\Gamma(X, \mathscr{C}^p(\mathfrak{U}, \mathscr{F})) = C^p(\mathfrak{U}, \mathscr{F}).$$

Lemma 21 Resolution by Čech Sheaves.

There is a natural augmentation map $\varepsilon : \mathscr{F} \to \mathscr{C}^0(\mathfrak{U}, \mathscr{F})$. The resulting sequence

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varepsilon} \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \xrightarrow{d^0} \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \xrightarrow{d^1} \ldots$$

is an **exact sequence** of sheaves. Thus, $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ provides a resolution of $\mathscr{F}$ (although not necessarily an injective or flasque one, unless specific conditions are met).

Proof. To prove the exactness of a sequence of sheaves, it suffices to verify exactness at the level of stalks for every point $x \in X$.

Step 1: Exactness at $\mathscr{F}$ (Injectivity). The map ε is defined by restricting a section to each U_i. The exactness at the start, $0 \to \mathscr{F} \to \mathscr{C}^0 \to \mathscr{C}^1$, is equivalent to the statement that $\mathscr{F}$ is a sheaf (the first two sheaf axioms), which implies the sequence is exact.

Step 2: Exactness at $\mathscr{C}^p$ for $p \geq 1$ (Homotopy Argument). Fix a point $x \in X$. Since $\mathfrak{U}$ covers X, there exists at least one index $k \in I$ such that $x \in U_k$. This specific index k allows us to construct a **contracting homotopy** on the complex of stalks $(\mathscr{C}_x^\bullet, d_x)$.

Define a map $h : \mathscr{C}_x^p \to \mathscr{C}_x^{p-1}$ as follows: for a germ $\alpha \in \mathscr{C}_x^p$, represented by a section over a small neighborhood $V \subset U_k$, the component of $h(\alpha)$ is given by "adding the index k to the front":

$$(h\alpha)_{i_0 \dots i_{p-1}} = \alpha_{k i_0 \dots i_{p-1}}.$$

(Note: If the indices become unordered, we use the standard alternating convention to reorder them).

We now verify the homotopy relation $dh + hd = \mathrm{id}$. Recall the boundary formula: $(d\alpha)_{j_0 \dots j_{p+1}} = \sum (-1)^m \alpha_{\dots \hat{j}_m \dots}$. When we compute $(h(d\alpha))_{i_0 \dots i_p}$, we insert k at the beginning, getting $(d\alpha)_{k i_0 \dots i_p}$. This sum contains two types of terms:

- The term where index k is removed: this returns $\alpha_{i_0 \dots i_p}$.
- The terms where one of the i_m is removed: these correspond to $-(d(h\alpha))_{i_0 \dots i_p}$.

Precisely, one can calculate that $(dh + hd)(\alpha) = \alpha$.

Since the identity map on the complex of stalks is homotopic to the zero map (via h), the cohomology of the stalk complex is zero for all $p \geq 1$. This implies the sequence is exact. □

Proposition 22 Vanishing of Čech Cohomology for Flasque Sheaves.

Let X be a topological space and $\mathfrak{U}$ an open covering. If $\mathscr{F}$ is a flasque sheaf on X, then for all $p > 0$:

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) = 0.$$

Proof. Recall the Čech sheaf resolution constructed in **Lemma 21**:

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varepsilon} \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \xrightarrow{d^0} \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \dots$$

We examine the terms of this complex. The sheaf $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is defined as a product of direct images:

$$\mathscr{C}^p(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < \dots < i_p} (j_{i_0 \dots i_p})_* \left(\mathscr{F}|_{U_{i_0 \dots i_p}} \right).$$

Since $\mathscr{F}$ is flasque, its restriction to any open set is flasque. By **Lemma 10**, the direct image of a flasque sheaf is flasque. Furthermore, the product of flasque sheaves is flasque. Therefore, each term $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is a flasque sheaf.

The sequence above is an exact sequence of sheaves. Since $\mathscr{F}$ is flasque, and each $\mathscr{C}^p$ is flasque, the sequence of global sections remains exact (by repeated application of **Lemma 10(2)** to the kernels of the maps). The complex of global sections is precisely the standard Čech complex:

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow C^0(\mathfrak{U}, \mathscr{F}) \longrightarrow C^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \dots$$

Since this sequence is exact, the cohomology groups of the complex $C^\bullet(\mathfrak{U}, \mathscr{F})$ vanish for $p > 0$. Thus $\check{H}^p(\mathfrak{U}, \mathscr{F}) = 0$. □

Lemma 23 Natural Morphism from Čech to Derived Cohomology.

Let X be a topological space and $\mathfrak{U}$ an open covering. For any sheaf $\mathscr{F}$ and all $p \geq 0$, there exists a natural homomorphism:

$$\theta^p : \check{H}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow H^p(X, \mathscr{F}).$$

This family of maps is functorial in $\mathscr{F}$.

Proof. We compare two resolutions of $\mathscr{F}$.

1. The Čech sheaf resolution: $0 \to \mathscr{F} \to \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$.
2. An injective resolution: $0 \to \mathscr{F} \to \mathscr{I}^\bullet$.

By the fundamental lemma of homological algebra, since $\mathscr{C}^\bullet$ is a resolution and $\mathscr{I}^\bullet$ consists of injective objects (which allow lifting of morphisms), the identity map $\mathrm{id} : \mathscr{F} \to \mathscr{F}$ extends uniquely (up to homotopy) to a chain map of complexes:

$$\psi^\bullet : \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{I}^\bullet.$$

Applying the global section functor $\Gamma(X, \cdot)$, we obtain a map of complexes of abelian groups:

$$\Gamma(\psi^\bullet) : C^\bullet(\mathfrak{U}, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{I}^\bullet).$$

Passing to cohomology, this induces the desired map $\theta^p : \check{H}^p(\mathfrak{U}, \mathscr{F}) \to H^p(X, \mathscr{F})$. For $p = 0$, we have established that both groups are isomorphic to $\Gamma(X, \mathscr{F})$, and θ^0 is indeed this canonical isomorphism. □

4.3 Cohomology of Sheaves on a Noetherian Affine Scheme

Lemma 24 Exactness of Global Sections on Affine Schemes.

Let $X = \mathrm{Spec} A$ be an affine scheme. Let

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

be a short exact sequence of **quasi-coherent** sheaves on X. Then the induced sequence of global sections (A-modules) is exact:

$$0 \longrightarrow \Gamma(X, \mathscr{F}') \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}'') \longrightarrow 0.$$

Proof. Since $X = \mathrm{Spec} A$ and the sheaves are quasi-coherent, they correspond to A-modules. Let $M' = \Gamma(X, \mathscr{F}')$, $M = \Gamma(X, \mathscr{F})$, and $M'' = \Gamma(X, \mathscr{F}'')$. Then $\mathscr{F}' \cong \widetilde{M'}$, $\mathscr{F} \cong \widetilde{M}$, and $\mathscr{F}'' \cong \widetilde{M''}$.

The left exactness of the global section functor is true for any sheaves. Thus, we have an exact sequence of A-modules:

$$0 \longrightarrow M' \longrightarrow M \xrightarrow{\psi} M''.$$

We only need to prove that ψ is surjective. Let $K = \mathrm{Coker}(\psi) = M''/\mathrm{Im}(\psi)$. We have an exact sequence of A-modules:

$$M \xrightarrow{\psi} M'' \longrightarrow K \longrightarrow 0.$$

Since the "tilde" functor $N \mapsto \widetilde{N}$ from A-modules to quasi-coherent sheaves is exact, applying it yields an exact sequence of sheaves:

$$\widetilde{M} \longrightarrow \widetilde{M''} \longrightarrow \widetilde{K} \longrightarrow 0.$$

This corresponds to the sequence $\mathscr{F} \to \mathscr{F}'' \to \widetilde{K} \to 0$. By the hypothesis, the map $\mathscr{F} \to \mathscr{F}''$ is surjective (as a map of sheaves), which implies that its cokernel is the zero sheaf. Therefore, $\widetilde{K} = 0$.

The sheaf $\widetilde{K}$ is zero if and only if for every prime ideal $\mathfrak{p} \in \mathrm{Spec}A$, the stalk $(\widetilde{K})_{\mathfrak{p}} = K_{\mathfrak{p}}$ is zero. A fundamental fact of commutative algebra states that if $K_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}A$ (or even just for all maximal ideals), then $K = 0$.

Since $K = 0$, the map $M \to M''$ is surjective. Thus, the sequence of global sections is exact. □

Proposition 25 Čech-Derived Isomorphism for Affine Schemes.

Let $X = \mathrm{Spec}A$ be a Noetherian affine scheme. Let $\mathscr{F}$ be a quasi-coherent sheaf on X. Let $\mathfrak{U} = (U_i)_{i\in I}$ be a finite open covering of X by affine open sets (e.g., basic open sets $D(f_i)$). Then the natural map is an isomorphism for all $p \geq 0$:

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) \xrightarrow{\sim} H^p(X, \mathscr{F}).$$

Proof. We proceed by induction on p. For $p = 0$, the result holds by **Lemma** 19.

For the inductive step, we assume the result holds for degree $p - 1$ for any quasi-coherent sheaf. We construct a "dimension shifting" argument using flasque embeddings. Since the category of quasi-coherent sheaves on a Noetherian affine scheme has enough injectives, we can embed $\mathscr{F}$ into a flasque quasi-coherent sheaf $\mathscr{G}$ (in fact, injective quasi-coherent sheaves are flasque). Let $\mathcal{R}$ be the quotient:

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathcal{R} \longrightarrow 0.$$

Here $\mathcal{R}$ is also quasi-coherent.

Consider the Čech complex associated with this sequence. For any intersection of affine open sets $U_{i_0\ldots i_k} = U_{i_0} \cap \cdots \cap U_{i_k}$, the restriction of the sequence is:

$$0 \longrightarrow \mathscr{F}(U_{i_0\ldots i_k}) \longrightarrow \mathscr{G}(U_{i_0\ldots i_k}) \longrightarrow \mathcal{R}(U_{i_0\ldots i_k}) \longrightarrow 0.$$

Crucially, since X is affine (and separated), the intersection of affine opens is affine. On an affine scheme, the functor of global sections is exact for quasi-coherent sheaves (a property equivalent to $H^1 = 0$, or Serre's Theorem B). Thus, the sequence of sections above is exact. Taking the product over all indices, we obtain a short exact sequence of Čech complexes:

$$0 \longrightarrow C^\bullet(\mathfrak{U}, \mathscr{F}) \longrightarrow C^\bullet(\mathfrak{U}, \mathscr{G}) \longrightarrow C^\bullet(\mathfrak{U}, \mathcal{R}) \longrightarrow 0.$$

This induces a long exact sequence in Čech cohomology:

$$\cdots \to \check{H}^{p-1}(\mathfrak{U}, \mathscr{G}) \to \check{H}^{p-1}(\mathfrak{U}, \mathcal{R}) \xrightarrow{\delta} \check{H}^{p}(\mathfrak{U}, \mathscr{F}) \to \check{H}^{p}(\mathfrak{U}, \mathscr{G}) \to \dots$$

Since $\mathscr{G}$ is flasque, $\check{H}^k(\mathfrak{U}, \mathscr{G}) = 0$ for all $k > 0$ (**Proposition** 22). Thus, for $p \geq 2$, we have an isomorphism $\check{H}^{p-1}(\mathfrak{U}, \mathcal{R}) \cong \check{H}^{p}(\mathfrak{U}, \mathscr{F})$. For $p = 1$, we use the cokernel of the 0-th level map.

Similarly, for the derived functor cohomology, the short exact sequence of sheaves yields a long exact sequence. Since $\mathscr{G}$ is flasque (and thus acyclic), $H^k(X, \mathscr{G}) = 0$ for $k > 0$. This yields an isomorphism $H^{p-1}(X, \mathcal{R}) \cong H^p(X, \mathscr{F})$.

By the induction hypothesis, $\check{H}^{p-1}(\mathfrak{U}, \mathcal{R}) \cong H^{p-1}(X, \mathcal{R})$. Combining the isomorphisms from both sides, we conclude that $\check{H}^{p}(\mathfrak{U}, \mathscr{F}) \cong H^p(X, \mathscr{F})$. □

Note 26 Generalization to Separated Schemes.

The only property of affine schemes used in the proof above is that the intersection of two affine open sets is again affine. This property holds for any **separated scheme**. Thus, the isomorphism $\check{H}^{p}(\mathfrak{U}, \mathscr{F}) \cong H^p(X, \mathscr{F})$ holds for any Noetherian separated scheme, provided $\mathfrak{U}$ is an affine covering and $\mathscr{F}$ is quasi-coherent. This allows us to compute sheaf cohomology entirely using the combinatorics of affine rings.

Theorem 27 Serre's Theorem B (Affine Vanishing).

Let $X = \operatorname{Spec} A$ be a Noetherian affine scheme, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then for all $p > 0$, the cohomology group $H^p(X, \mathscr{F}) = 0$.

Proof 1: Algebraic Cech Cohomology. We compute the cohomology using the Čech complex associated to a finite affine open covering $\mathcal{U} = \{D(f_i)\}_{i=0}^n$ of X. Since $\mathscr{F}$ is quasi-coherent, $\mathscr{F} = \widetilde{M}$ for some A-module M.

1. The Čech Complex The p-th term of the Čech complex is given by the product of localizations:

$$C^p(\mathcal{U}, M) = \prod_{0 \leq i_0 < \cdots < i_p \leq n} M_{f_{i_0} \cdots f_{i_p}}.$$

An element $\alpha \in C^p$ is determined by its components $\alpha_{i_0 \dots i_p} \in M_{f_{i_0} \cdots f_{i_p}}$.

2. Partition of Unity Since $\mathcal{U}$ covers $\operatorname{Spec} A$, the ideal generated by $f_0, \dots, f_n$ is the unit ideal (1). Thus, there exist $a_0, \dots, a_n \in A$ such that:

$$\sum_{k=0}^{n} a_k f_k = 1.$$

This algebraic identity is the engine that drives the global vanishing of cohomology.

3. Construction of the Homotopy Operator To prove that the complex is exact for $p > 0$, we construct a contracting homotopy $h : C^p \to C^{p-1}$ such that $dh + hd = \mathrm{id}$.

We define the operator h using a weighted sum of "local" homotopies. For $\alpha \in C^p$, we define the component of $h\alpha$ indexed by $i_0 \dots i_{p-1}$ as:

$$(h\alpha)_{i_0 \dots i_{p-1}} = \sum_{k=0}^{n} a_k f_k \cdot \alpha_{k i_0 \dots i_{p-1}}.$$

Note: In this formula, the index set of α is $\{k, i_0, \dots, i_{p-1}\}$. We adopt the standard convention that if the indices are not sorted, we permute them to sorted order and multiply by the sign of the permutation. If k is already present in $\{i_0, \dots, i_{p-1}\}$, the term is zero (alternating property). The multiplication by f_k ensures compatibility with the localization maps (effectively mapping $M_{f_k f_I} \to M_{f_I}$ in the context of the resolution).

4. Verification of the Identity We verify $(dh + hd)(\alpha) = \alpha$.

First, compute $(d(h\alpha))_{j_0 \dots j_p}$:

$$(d(h\alpha))_J = \sum_{r=0}^{p} (-1)^r (h\alpha)_{j_0 \dots \hat{j}_r \dots j_p} = \sum_{r=0}^{p} (-1)^r \sum_{k=0}^{n} a_k f_k \alpha_{k j_0 \dots \hat{j}_r \dots j_p}.$$

Second, compute $(h(d\alpha))_{j_0 \dots j_p}$:

$$(h(d\alpha))_J = \sum_{k=0}^{n} a_k f_k (d\alpha)_{k j_0 \dots j_p}.$$

Expanding the boundary term $(d\alpha)_{k j_0 \dots j_p}$:

$$(d\alpha)_{k j_0 \dots j_p} = \alpha_{j_0 \dots j_p} - \sum_{r=0}^{p} (-1)^r \alpha_{k j_0 \dots \hat{j}_r \dots j_p}.$$

(Here, the first term comes from omitting the first index k, and the sum comes from omitting indices from J).

Now, sum the two results:

$$\begin{aligned}
(dh + hd)(\alpha)_J &= \sum_{k=0}^{n} a_k f_k \left[\sum_{r=0}^{p} (-1)^r \alpha_{k j_0 \dots \hat{j}_r \dots j_p} + \alpha_J - \sum_{r=0}^{p} (-1)^r \alpha_{k j_0 \dots \hat{j}_r \dots j_p} \right] \\
&= \sum_{k=0}^{n} a_k f_k \cdot \alpha_J \\
&= \left(\sum_{k=0}^{n} a_k f_k \right) \cdot \alpha_J \\
&= 1 \cdot \alpha_J = \alpha_J.
\end{aligned}$$

The "cross terms" (where we omit an index from J) cancel out perfectly due to the alternating signs. The only surviving term corresponds to omitting the newly added index k from the boundary of the extended set, which yields α_J scaled by the partition of unity.

Since the identity map is homotopic to the zero map, the cohomology vanishes for all $p > 0$. □

Proof 2: Categorical Proof via Exactness of Global Sections. This proof relies on the foundational properties of the category of quasi-coherent sheaves on affine schemes.

1. Equivalence of Categories For an affine scheme $X = \mathrm{Spec} A$, the functor $\mathscr{F} \mapsto \Gamma(X, \mathscr{F})$ provides an equivalence of categories between $\mathfrak{QCoh}(X)$ (quasi-coherent sheaves) and $\mathsf{Mod}(A)$ (A-modules). The inverse functor is the "tilde" construction $M \mapsto \widetilde{M}$.

2. Exactness of Global Sections In the category of modules $\mathsf{Mod}(A)$, a sequence $0 \to M' \to M \to M'' \to 0$ is exact if and only if it is exact as a sequence of sets/abelian groups. Under the equivalence of categories, if

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

is a short exact sequence of quasi-coherent sheaves, applying the global section functor $\Gamma(X, -)$ yields a sequence of A-modules

$$0 \longrightarrow \Gamma(X, \mathscr{F}') \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}'') \longrightarrow 0.$$

This sequence corresponds exactly to the module sequence $0 \to M' \to M \to M'' \to 0$, which is exact. Thus, on an affine scheme, the functor $\Gamma(X, -)$ is **exact** on the category of quasi-coherent sheaves.

3. Vanishing of Derived Functors The cohomology groups $H^p(X, -)$ are defined as the right derived functors of $\Gamma(X, -)$. A general theorem of homological algebra states that if a functor is exact, its higher derived functors are zero.

More formally, we can compute derived functors using an injective resolution. Since $\Gamma(X, -)$ preserves exactness, applying it to an exact resolution $0 \to \mathscr{F} \to \mathscr{I}^0 \to \mathscr{I}^1 \to \dots$ yields an exact sequence of global sections. The cohomology of an exact sequence is zero everywhere except at degree 0. Therefore, $H^p(X, \mathscr{F}) = 0$ for all $p > 0$. ☐

4.4 Examples

Example 28 Singular Cohomology as Sheaf Cohomology.

Let X be a topological space and let G be an abelian group. We consider the relationship between the classical **singular cohomology** $H^n_{\mathrm{sing}}(X, G)$ and the **sheaf cohomology** $H^n(X, \underline{G})$ with coefficients in the constant sheaf $\underline{G}$.

Theorem: The Comparison Isomorphism. If X is a **locally contractible** space (e.g., a topological manifold or a CW complex), then there is a natural isomorphism for all $n \geq 0$:

$$H^n_{\mathrm{sing}}(X, G) \cong H^n(X, \underline{G}).$$

Proof Strategy: Resolution by Singular Cochains. The proof relies on interpreting singular cochains locally to form a resolution of the constant sheaf. Let $U \subseteq X$ be an open set. Let $C^n(U, G)$ denote the group of singular n-cochains on U with coefficients in G. The assignment $U \mapsto C^n(U, G)$ defines a presheaf. Let $\mathcal{S}^n$ be the sheafification of this presheaf, called the **sheaf of local singular n-cochains**.

We construct a sequence of sheaves:

$$0 \longrightarrow \underline{G} \xrightarrow{\varepsilon} \mathcal{S}^0 \xrightarrow{d} \mathcal{S}^1 \xrightarrow{d} \mathcal{S}^2 \longrightarrow \dots$$

where ε embeds the constant group as locally constant functions (0-cochains), and d is induced by the singular coboundary operator.

Step 1: Exactness (The Local Lemma). We must show this sequence is exact as a complex of sheaves. Exactness is a local property. Since X is locally contractible, every point x has a neighborhood base consisting of contractible open sets U_α. On a contractible set U_α, the singular cohomology vanishes in positive degrees (Poincaré Lemma for singular cohomology): $H^n_{\text{sing}}(U_\alpha, G) = 0$ for $n > 0$, and $H^0_{\text{sing}}(U_\alpha, G) = G$. This implies that the sequence of sections

$$0 \to G \to C^0(U_\alpha, G) \to C^1(U_\alpha, G) \to \dots$$

is exact. Passing to the direct limit over neighborhoods, we see that the sequence of stalks at every point is exact. Thus, $\mathcal{S}^\bullet$ is a resolution of $\underline{G}$.

Step 2: Acyclicity of the Resolution. The sheaves $\mathcal{S}^n$ are "flabby" (flasque) or sufficiently fine (soft). Specifically, singular cochains can be extended from open sets (unlike differential forms, singular cochains are just functions on the set of simplices, so extension is often trivial if handled set-theoretically, though technical care is needed with "small" simplices in the sheafification process). Assuming appropriate paracompactness or using the precise definition of $\mathcal{S}^n$, these sheaves satisfy $H^q(X, \mathcal{S}^n) = 0$ for $q > 0$.

Conclusion. Since $\mathcal{S}^\bullet$ is an acyclic resolution of $\underline{G}$, the cohomology of the sheaf $\underline{G}$ is computed by the cohomology of the complex of global sections $\Gamma(X, \mathcal{S}^\bullet)$. For "nice" spaces (paracompact and locally contractible), the global sections $\Gamma(X, \mathcal{S}^n)$ coincide with the group of global singular cochains $C^n(X, G)$ (modulo technicalities about "small" simplices, which are handled by the excision axiom). Therefore,

$$H^n(X, \underline{G}) \cong h^n(\Gamma(X, \mathcal{S}^\bullet)) \cong H^n_{\text{sing}}(X, G).$$

This identifies the abstract derived functor with the geometrically defined topological invariant.

Example 29 Contrast: Smooth vs. Holomorphic Functions.

Let M be a manifold. We compare the cohomology of the structure sheaves in the smooth and complex categories. This comparison illustrates the fundamental difference between "flexible" and "rigid" geometry.

1. The Smooth Case (Flexible) Let M be a paracompact **smooth (C^∞) manifold**. Let C^∞_M denote the sheaf of smooth real-valued functions on M.

- **Partitions of Unity:** The defining feature of smooth manifolds is the existence of partitions of unity. For any open cover $\mathfrak{U}$, we can find smooth functions supported in the open sets summing to 1.
- **Fine Sheaves:** A sheaf admitting partitions of unity is called a **fine sheaf**. A fundamental theorem states that fine sheaves are **acyclic**.

- **Cohomology:** Consequently, for all $p > 0$:

$$H^p(M, C^\infty_M) = 0.$$

Interpretation: There are no global topological obstructions to solving linear differential equations (like the $\bar{\partial}$-problem) locally in the smooth category, provided we allow solutions to be merely smooth. The global objects can always be glued from local ones using bump functions. The "geometry" of M is not reflected in the higher cohomology of its structure sheaf.

2. The Holomorphic Case (Rigid) Let X be a **complex manifold**. Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions.

- **No Partitions of Unity:** Holomorphic functions are extremely rigid. By the Identity Theorem, a holomorphic function cannot be zero on an open set without being zero everywhere on that connected component. Thus, we cannot construct "holomorphic bump functions," and partitions of unity do not exist in the holomorphic category.
- **Non-Trivial Cohomology:** Consequently, $\mathcal{O}_X$ is generally **not acyclic**. The groups $H^p(X, \mathcal{O}_X)$ are non-zero and are crucial geometric invariants.
- **Example:** If X is a compact Riemann surface of genus g, then:

$$H^0(X, \mathcal{O}_X) \cong \mathbb{C} \quad \text{(Liouville's Theorem)},$$

$$H^1(X, \mathcal{O}_X) \cong \mathbb{C}^g \quad \text{(related to the space of differential forms)}.$$

Interpretation: The non-vanishing of $H^1(X, \mathcal{O}_X)$ measures the obstruction to solving the $\bar{\partial}$-equation globally. It reflects the rigidity of the complex structure.

3. The Connection: Dolbeault Cohomology The relationship between these two worlds is bridged by the **Dolbeault Resolution**. Although $\mathcal{O}_X$ is not acyclic, it admits a resolution by fine sheaves (the sheaves of smooth differential forms):

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{A}^{0,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,2} \longrightarrow \dots$$

Since the sheaves $\mathcal{A}^{0,q}$ (smooth forms) are fine, their higher cohomology vanishes. By the abstract de Rham theorem, the cohomology of $\mathcal{O}_X$ is isomorphic to the cohomology of the complex of global sections:

$$H^p(X, \mathcal{O}_X) \cong H^{0,p}_{\bar{\partial}}(X) = \frac{\operatorname{Ker}(\bar{\partial} : \mathcal{A}^{0,p}(X) \to \mathcal{A}^{0,p+1}(X))}{\operatorname{Im}(\bar{\partial})}.$$

This is the celebrated **Dolbeault Isomorphism**, expressing the sheaf cohomology of the rigid sheaf $\mathcal{O}_X$ in terms of differential forms defined via the flexible smooth structure.

Example 30 De Rham Cohomology and Sheaf Cohomology.

Let M be a smooth (paracompact) manifold. We aim to establish the isomorphism between the classical De Rham cohomology $H^k_{\mathrm{dR}}(M)$ and the sheaf cohomology $H^k(M, \underline{\mathbb{R}})$ with coefficients in the constant sheaf.

1. The De Rham Complex of Sheaves. Let $\mathcal{A}^k$ denote the sheaf of smooth differential k-forms on M. The exterior derivative d induces a morphism of sheaves $d : \mathcal{A}^k \to \mathcal{A}^{k+1}$. We consider the sequence of sheaves augmented by the constant sheaf $\underline{\mathbb{R}}$:

$$0 \longrightarrow \underline{\mathbb{R}} \xrightarrow{\varepsilon} \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \xrightarrow{d} \dots$$

Here, ε is the inclusion of locally constant functions into smooth functions (viewed as 0-forms).

2. Exactness (The Poincaré Lemma). To prove that this sequence is a resolution of $\underline{\mathbb{R}}$, we must verify exactness at the level of stalks. Let $x \in M$. For any small neighborhood U of x which is contractible (e.g., a star-shaped open set diffeomorphic to a ball), the classical **Poincaré Lemma** states that every closed form is exact. Specifically, if $\omega \in \mathcal{A}^k(U)$ satisfies $d\omega = 0$ with $k > 0$, there exists $\eta \in \mathcal{A}^{k-1}(U)$ such that $d\eta = \omega$. For $k = 0$, the kernel of $d : \mathcal{A}^0(U) \to \mathcal{A}^1(U)$ consists of functions with zero gradient, i.e., locally constant functions. Taking the direct limit over such neighborhoods, we conclude that the sequence of stalks is exact at every point. Thus, $0 \to \underline{\mathbb{R}} \to \mathcal{A}^\bullet$ is an exact resolution.

3. Acyclicity (Fine Sheaves). The sheaves $\mathcal{A}^k$ are sheaves of modules over the sheaf of smooth rings C^∞_M. Since M is a smooth paracompact manifold, it admits **partitions of unity**. This allows us to "glue" local sections arbitrarily: for any locally finite cover $\{U_i\}$, we can construct global forms using bump functions. Consequently, the sheaves $\mathcal{A}^k$ are **fine sheaves** (a stronger condition than soft or flasque). A fundamental result in sheaf theory asserts that fine sheaves are acyclic, meaning $H^q(M, \mathcal{A}^k) = 0$ for all $q > 0$.

4. The Isomorphism. Since $\mathcal{A}^\bullet$ is an acyclic resolution of $\underline{\mathbb{R}}$, the abstract De Rham Theorem (the general comparison theorem for derived functors) implies that the cohomology of the sheaf $\underline{\mathbb{R}}$ is isomorphic to the cohomology of the complex of global sections $\Gamma(M, \mathcal{A}^\bullet)$. The complex of global sections is precisely the classical De Rham complex of the manifold:

$$\Gamma(M, \mathcal{A}^\bullet) = \big(\Omega^\bullet(M), d\big).$$

Therefore, we obtain the canonical isomorphism:

$$H^k(M, \underline{\mathbb{R}}) \cong h^k(\Omega^\bullet(M)) = H^k_{\mathrm{dR}}(M).$$

CHAPTER 5

General Schemes and Cohomology

5.1 Basic Definition of Schemes

We now assemble the pieces to define the central object of algebraic geometry: the scheme. A scheme is a topological space equipped with a sheaf of rings that locally looks like the spectrum of a ring.

We have already studied the model objects: affine schemes. Since schemes are locally affine, we can construct general schemes by gluing affine schemes together. Instead of treating topological gluing and sheaf gluing separately, we formulate a unified "Gluing Lemma" in the category of locally ringed spaces.

We now assert that these data uniquely determine a global object.

Theorem 1 Gluing of Locally Ringed Spaces.

Given a gluing datum $(\{X_i\}, \{U_{ij}\}, \{\varphi_{ij}\})$ in the category of locally ringed spaces, there exists a locally ringed space $(X, \mathcal{O}_X)$ together with morphisms $\psi_i : X_i \to X$, satisfying:

1. ψ_i is an isomorphism from X_i onto an open subspace of X.
2. The open sets cover X, i.e., $X = \bigcup_{i \in I} \psi_i(X_i)$.
3. The morphisms are compatible with the gluing maps: $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}.

Furthermore, X satisfies the following **Universal Property**: For any locally ringed space Y and a family of morphisms $\{f_i : X_i \to Y\}$ such that $f_i|_{U_{ij}} = f_j|_{U_{ji}} \circ \varphi_{ij}$, there exists a unique morphism $f : X \to Y$ making the diagram commute for all i.

Proof. **Step 1: The Topological Space.** Let $X = \varinjlim_{i \in I} X_i$ be the colimit. The cocycle conditions ensure $\sim$ is a valid equivalence relation. One verifies that the natural maps $\psi_i : X_i \to X$ are homeomorphisms onto open sets.

Step 2: The Structure Sheaf. We define $\mathcal{O}_X$ using the sheaf gluing property (or simply by definition on the quotient). For any open set $V \subseteq X$, a section $s \in \mathcal{O}_X(V)$ is a collection of sections $\{s_i \in \mathcal{O}_{X_i}(\psi_i^{-1}(V))\}$ such that for all i, j, the restriction of s_i to overlaps corresponds to s_j via the ring isomorphisms induced by φ_{ij}. Since sheaves are defined by local data, this forms a well-defined sheaf of rings.

Step 3: Locally Ringed Property. Since $\mathscr{O}_X|_{X_i} \cong \mathscr{O}_{X_i}$ and each $(X_i, \mathscr{O}_{X_i})$ is a locally ringed space, the stalks of $\mathscr{O}_X$ are local rings. Thus $(X, \mathscr{O}_X)$ is a locally ringed space. □

The universal property mentioned above is essentially the "Gluing Lemma for Morphisms." It can be restated as the sheaf property of the Hom-functor.

Proposition 2 Sheaf Property of Morphisms.

Let X and Y be locally ringed spaces (or schemes). Let $\{U_i\}$ be an open cover of X.

1. If $f, g : X \to Y$ are two morphisms such that $f|_{U_i} = g|_{U_i}$ for all i, then $f = g$.
2. If $\{f_i : U_i \to Y\}$ is a collection of morphisms satisfying $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all pairs, then there exists a unique morphism $f : X \to Y$ such that $f|_{U_i} = f_i$.

In categorical terms, the functor $h_Y(-) = \mathrm{Hom}_{\mathfrak{Sch}}(-, Y)$ is a sheaf on the Zariski topology of X.

With the machinery of affine schemes and the gluing lemma, we define the general concept efficiently.

Definition 3 Scheme.

A **scheme** is a locally ringed space $(X, \mathscr{O}_X)$ that is locally isomorphic to an affine scheme. That is, there exists an open cover $\{U_i\}_{i \in I}$ of X such that for each i, the locally ringed space $(U_i, \mathscr{O}_X|_{U_i})$ is isomorphic to $(\operatorname{Spec} A_i, \mathscr{O}_{\operatorname{Spec} A_i})$ for some ring A_i.

Note 4 Equivalent Definition of Schemes.

By above, a scheme can be equivalently defined as the result of gluing a family of affine schemes $\{\operatorname{Spec} A_i\}$ along open subschemes.

Definition 5 Morphism of Schemes.

A morphism of schemes is simply a morphism of locally ringed spaces.

We denote by Sch the category of schemes. The definition implies a full embedding of the category of affine schemes into Sch. Furthermore, the global section functor $\Gamma : \mathsf{Sch} \to \mathsf{Ring}^{op}$ has a right adjoint $\operatorname{Spec} : \mathsf{Ring}^{op} \to \mathsf{Sch}$, establishing the fundamental link between algebra and geometry.

We now establish the bridge between the category of rings and the category of locally ringed spaces (and thus schemes). The classical result stating that affine schemes are equivalent to rings is merely a consequence of a much more general adjunction.

Theorem 6 Adjunction between Global Sections and Spectrum.

Let LRS be the category of locally ringed spaces. There is an adjunction between the global section functor $\Gamma : \mathsf{LRS} \to \mathsf{Ring}^{\mathrm{op}}$ and the spectrum functor $\operatorname{Spec} : \mathsf{Ring}^{\mathrm{op}} \to \mathsf{LRS}$. Explicitly, for any locally ringed space X and any ring A, there is a natural bijection:

$$\Phi : \mathrm{Hom}_{\mathsf{LRS}}(X, \mathrm{Spec} A) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Ring}}(A, \Gamma(X, \mathscr{O}_X)).$$

Proof. We construct the map and its inverse.

Step 1: From Geometry to Algebra (Φ): Let $(f, f^\#) : X \to \operatorname{Spec} A$ be a morphism. The sheaf map $f^\# : \mathscr{O}_{\operatorname{Spec} A} \to f_*\mathscr{O}_X$ induces a map on global sections:

$$\Gamma(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}) \xrightarrow{f^\#(X)} \Gamma(\operatorname{Spec} A, f_*\mathscr{O}_X) = \Gamma(X, \mathscr{O}_X).$$

Since $\Gamma(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}) \cong A$, this gives a ring homomorphism $\phi : A \to \Gamma(X, \mathscr{O}_X)$. Thus, $\Phi(f) = \phi$.

Step 2: From Algebra to Geometry (Φ^{-1}): Let $\phi : A \to \Gamma(X, \mathscr{O}_X)$ be a ring homomorphism. We must construct a morphism $f : X \to \operatorname{Spec} A$.

Step 2a: The Topological Map. For any point $x \in X$, we have a canonical map from global sections to the residue field $\kappa(x)$:

$$\mathrm{ev}_x : \Gamma(X, \mathscr{O}_X) \longrightarrow \mathscr{O}_{X,x} \longrightarrow \kappa(x).$$

Composing with ϕ, we get a map $A \to \kappa(x)$. The kernel of this map is a prime ideal in A (since the target is a field). We define $f(x)$ to be this prime ideal:

$$f(x) := \ker(\mathrm{ev}_x \circ \phi) \in \operatorname{Spec} A.$$

It is a standard verification that this map is continuous (preimage of $D(a)$ is the open set where $\phi(a)$ is invertible).

Step 2b: The Sheaf Map. We need a map $f^\# : \mathscr{O}_{\operatorname{Spec} A} \to f_*\mathscr{O}_X$. Since $\mathscr{O}_{\operatorname{Spec} A}$ is defined on the basis $\{D(a)\}_{a \in A}$, it suffices to define compatible maps on this basis. On $D(a)$, we have $\mathscr{O}_{\operatorname{Spec} A}(D(a)) = A_a$. We need a map $A_a \to \Gamma(f^{-1}(D(a)), \mathscr{O}_X)$. Note that $f^{-1}(D(a)) = \{x \in X \mid \phi(a) \notin \mathfrak{m}_x\}$. This is exactly the locus where the section $\phi(a) \in \Gamma(X, \mathscr{O}_X)$ is invertible in the stalks. Thus $\phi(a)$ is a unit in $\Gamma(f^{-1}(D(a)), \mathscr{O}_X)$. Therefore, the universal property of localization induces a unique map:

$$A_a \longrightarrow \Gamma(f^{-1}(D(a)), \mathscr{O}_X).$$

These maps are compatible with restriction, defining a morphism of sheaves.

Step 2c: The Local Property. We check the stalks. Let $x \in X$ and $\mathfrak{p} = f(x)$. The map on stalks $f_x^\# : A_\mathfrak{p} \to \mathscr{O}_{X,x}$ is the localization of the composite $A \xrightarrow{\phi} \Gamma(X) \to \mathscr{O}_{X,x}$. By our definition of $f(x)$, the preimage of $\mathfrak{m}_x$ under this map is exactly $\mathfrak{p}$. Thus, elements in $A \setminus \mathfrak{p}$ map to units in $\mathscr{O}_{X,x}$, and the map is a local homomorphism.

The construction in Step 2 provides the inverse to Φ. □

Corollary 7 Equivalence of Affine Schemes and Rings.

The functor $\operatorname{Spec} : \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Sch}$ is fully faithful. Its image is the category of affine schemes. Thus:

$$\mathsf{AffSch} \cong \mathsf{Ring}^{\mathrm{op}}.$$

Proof. Apply the Adjunction Theorem with $X = \mathrm{Spec} B$.

$$\mathrm{Hom}_{\mathsf{Sch}}(\mathrm{Spec} B, \mathrm{Spec} A) \cong \mathrm{Hom}_{\mathsf{Ring}}(A, \Gamma(\mathrm{Spec} B, \mathscr{O})) \cong \mathrm{Hom}_{\mathsf{Ring}}(A, B).$$

This shows the functor is fully faithful. The essential image is, by definition, the affine schemes. □

With the fundamental adjunction $\mathrm{Hom}(X, \mathrm{Spec} A) \cong \mathrm{Hom}(A, \Gamma(X))$ and the rigorous construction of structure sheaves, the standard structural results for schemes follow immediately.

Proposition 8 Principal Open Sets in Affine Schemes are Affine.

Let $X = \mathrm{Spec} A$ and let $f \in A$. The open subspace $D(f)$, equipped with the restricted structure sheaf, is an affine scheme isomorphic to $\mathrm{Spec} A_f$.

$$\left(D(f), \mathscr{O}_X|_{D(f)}\right) \cong \left(\mathrm{Spec} A_f, \mathscr{O}_{\mathrm{Spec} A_f}\right).$$

Proof. By our construction of the structure sheaf $\mathscr{O}_X$ on the basis, the restriction of $\mathscr{O}_X$ to the basis element $D(f)$ is precisely the sheaf defined by the ring A_f. Since the topological space $D(f) \subset \mathrm{Spec} A$ is homeomorphic to $\mathrm{Spec} A_f$, and the sheaves are defined by the same data on the basis, they are isomorphic as locally ringed spaces. □

Corollary 9 Open Subschemes.

Let X be a scheme and $U \subseteq X$ an open subset. Then $(U, \mathscr{O}_X|_U)$ is a scheme.

Proof. By definition, X has an open covering by affine schemes $X = \bigcup U_i$ where $U_i \cong \mathrm{Spec} A_i$. Then $U = \bigcup (U \cap U_i)$. The intersection $U \cap U_i$ is an open subset of the affine scheme $\mathrm{Spec} A_i$. By the basis property, $U \cap U_i$ can be covered by principal open sets $D(f_{ij}) \subseteq \mathrm{Spec} A_i$. By the Proposition above, each $D(f_{ij})$ is an affine scheme. Thus U is covered by affine schemes, so it is a scheme. □

Proposition 10 Morphisms to Affine Schemes.

Let X be **any** scheme (not necessarily affine) and A a ring. There is a canonical bijection:

$$\mathrm{Hom}_{\mathsf{Sch}}(X, \mathrm{Spec} A) \cong \mathrm{Hom}_{\mathsf{Ring}}(A, \Gamma(X, \mathscr{O}_X)).$$

Proof. This is exactly 6 we proved in the previous section for arbitrary locally ringed spaces. Since schemes are locally ringed spaces, the result holds. □

Definition 11 Quasi-coherent and Coherent Sheaves.

Let X be a scheme. A sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ is quasi-coherent if there exists an affine open covering $X = \bigcup U_i$, where $U_i = \mathrm{Spec}(A_i)$, such that for each i, the restriction of $\mathscr{F}$ to U_i is isomorphic to the sheaf associated to some A_i-module M_i:

$$\mathscr{F}|_{U_i} \cong \widetilde{M_i}$$

A sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ is coherent if:

i. $\mathscr{F}$ is of finite type, i.e., locally there exists a surjection $\mathscr{O}_X^n \to \mathscr{F}$.

ii. For any open set $U \subseteq X$ and any morphism $\phi : \mathscr{O}_U^n \to \mathscr{F}|_U$, the kernel $\ker(\phi)$ is also of finite type.

5.2 The Projective Space

We have defined affine schemes. To construct more interesting geometries (like projective spaces), we need a way to build global schemes by gluing together local affine "charts", much like constructing a manifold from coordinate charts.

We now rigorously construct the projective space $\mathbb{P}_A^n$ over a ring A by gluing $n+1$ copies of affine space $\mathbb{A}_A^n$. This construction mimics the definition of a manifold via coordinate charts.

Let $I = \{0, 1, \dots, n\}$. For each $i \in I$, we define the i-th chart U_i to be the affine scheme:

$$U_i := \operatorname{Spec} A[x_{i,0}, x_{i,1}, \dots, \hat{x}_{i,i}, \dots, x_{i,n}].$$

Here, the symbol $\hat{x}_{i,i}$ indicates that the variable $x_{i,i}$ is omitted. Thus, the ring has n variables over A. Geometrically, $U_i \cong \mathbb{A}_A^n$.

i. You should think of the variable $x_{i,j}$ as the ratio of homogeneous coordinates z_j/z_i on the region where $z_i \neq 0$.

ii. For convenience in calculations, we may set $x_{i,i} = 1$ in the fraction field, though it is not a generator of the polynomial ring.

We glue U_i and U_j along open subsets where the coordinate transformation is valid.

i. In the chart U_i, the overlap with U_j corresponds to the locus where the "coordinate" $x_{i,j}$ (representing z_j/z_i) is invertible.

$$U_{ij} := D(x_{i,j}) \subseteq U_i.$$

The ring of functions on U_{ij} is the localization $A[\dots, x_{i,k}, \dots]_{x_{i,j}}$.

ii. We define the isomorphism $\phi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$ by defining the corresponding ring isomorphism $\phi_{ij}^\#$:

$$\Gamma(U_{ji}, \mathscr{O}) \longrightarrow \Gamma(U_{ij}, \mathscr{O}).$$

The target ring is generated by $x_{i,k}$ and $x_{i,j}^{-1}$. The source ring is generated by $x_{j,k}$ and $x_{j,i}^{-1}$. The algebraic map is given by the formula (representing $\frac{z_k}{z_j} = \frac{z_k/z_i}{z_j/z_i}$):

$$\phi_{ij}^\#(x_{j,k}) := \frac{x_{i,k}}{x_{i,j}} \quad (\text{for } k \neq i, j), \qquad \phi_{ij}^\#(x_{j,i}) := \frac{1}{x_{i,j}}.$$

This is a well-defined homomorphism because $x_{i,j}$ is invertible in the domain U_{ij}. It is an isomorphism because $\phi_{ji}^\#$ provides the inverse.

To ensure these charts glue into a single space, we check the triple overlap condition on $U_{ijk} = U_i \cap U_j \cap U_k$. This reduces to checking the compatibility of ring maps:

$$\phi_{ik}^{\#} = \phi_{ij}^{\#} \circ \phi_{jk}^{\#}.$$

Let's track the image of a generator $x_{k,l}$ (representing z_l/z_k):

$$\begin{aligned}(\phi_{ij}^{\#} \circ \phi_{jk}^{\#})(x_{k,l}) &= \phi_{ij}^{\#}\left(\frac{x_{j,l}}{x_{j,k}}\right) \\ &= \frac{x_{i,l}/x_{i,j}}{x_{i,k}/x_{i,j}} \\ &= \frac{x_{i,l}}{x_{i,k}} = \phi_{ik}^{\#}(x_{k,l}).\end{aligned}$$

The algebraic identity holds. Thus, by the Gluing Lemma, there exists a unique scheme X, denoted $\mathbb{P}_A^n$.

Now that $X = \mathbb{P}_A^n$ is constructed, we analyze its structure using the tools we have developed.

Is it a Scheme? Yes. By the construction, X is covered by open sets isomorphic to U_i. Since each $U_i \cong$ Spec(Poly Ring) is an affine scheme, X is locally affine. Thus, $\mathbb{P}_A^n$ is a scheme.

Stalks: Let $P \in X$ be a point. Since $\{U_i\}$ covers X, P must lie in some chart U_i. The stalk $\mathscr{O}_{X,P}$ is isomorphic to the stalk of the structure sheaf of the affine scheme U_i at the point corresponding to P.

$$\mathscr{O}_{X,P} \cong (A[x_{i,0}, \dots, \hat{x}_{i,i}, \dots, x_{i,n}])_{\mathfrak{p}}.$$

This is a local ring of a polynomial ring. If A is a field, this is a regular local ring of dimension n.

Global Sections (A First Glimpse of "Compactness"): What are the global functions on projective space? Let $s \in \Gamma(X, \mathscr{O}_X)$. By definition, s corresponds to a collection of polynomials $f_i \in A[x_{i,0}, \dots]$ for each i, such that they agree on overlaps:

$$f_j = \phi_{ij}^{\#}(f_i) \quad \text{in } A[\dots, x_{j,k}, \dots]_{x_{j,i}}.$$

Consider the case $n = 1$ for simplicity. $U_0 = \mathrm{Spec} A[t]$, $U_1 = \mathrm{Spec} A[u]$. The overlap is given by $u = 1/t$. A global section is a pair $(f(t), g(u))$ such that $g(u) = f(1/u)$. Let $f(t) = a_0 + a_1 t + \cdots + a_d t^d$. Then

$$g(u) = a_0 + a_1 \frac{1}{u} + \cdots + a_d \frac{1}{u^d}.$$

For $g(u)$ to be a **polynomial** in u (no negative powers), we must have $a_1 = \cdots = a_d = 0$. Thus $f(t) = a_0$ is a constant. This generalizes to any n.

i. $\Gamma(\mathbb{P}_A^n, \mathscr{O}) \cong A$.

ii. This shows $\mathbb{P}_A^n$ is fundamentally different from $\mathbb{A}_A^n$ (whose global sections are polynomials). The only global regular functions on projective space are constants (analogous to Liouville's theorem for compact complex manifolds).

Theorem 12 Segre Embedding.

Let $\mathbb{P}^n$ and $\mathbb{P}^m$ be projective spaces over an algebraically closed field k. Let the homogeneous coordinates be $[x_0 : \cdots : x_n]$ and $[y_0 : \cdots : y_m]$ respectively.

The **Segre Map** is defined as:

$$\psi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N, \quad N = (n+1)(m+1) - 1$$

sending a pair of points to the point with coordinates $z_{i,j} = x_i y_j$, ordered lexicographically:

$$\psi([x_i], [y_j]) = [x_0 y_0 : x_0 y_1 : \cdots : x_i y_j : \cdots : x_n y_m]$$

Proof. **1. Well-definedness:** Since the coordinates of points in projective space are not all zero, there exist indices i, j such that $x_i \neq 0$ and $y_j \neq 0$. Thus, $x_i y_j \neq 0$. If we replace x_i with λx_i and y_j with μy_j, the new coordinates become $(\lambda\mu)x_i y_j$, which represents the same point in $\mathbb{P}^N$.

2. The Image as an Algebraic Variety: Let $Z_{i,j}$ be the coordinates on $\mathbb{P}^N$. We claim that the image $\Sigma_{n,m} = \psi(\mathbb{P}^n \times \mathbb{P}^m)$ is the zero locus of the quadratic polynomials:

$$S = \{Z_{i,j}Z_{k,l} - Z_{i,l}Z_{k,j} = 0 \mid 0 \leq i, k \leq n,\ 0 \leq j, l \leq m\}$$

Physically, this means the $(n+1) \times (m+1)$ matrix formed by $Z_{i,j}$ has rank 1. If $Z_{i,j} = x_i y_j$, then $x_i y_j x_k y_l - x_i y_l x_k y_j = 0$ is satisfied for all indices. Thus, $\mathrm{Im}(\psi) \subseteq V(S)$.

3. Injectivity and Inverse: Suppose $[z_{i,j}] \in V(S)$. At least one coordinate $z_{a,b}$ is non-zero. For any i, j, the equations $z_{i,j}z_{a,b} = z_{i,b}z_{a,j}$ imply:

$$z_{i,j} = \frac{z_{i,b}z_{a,j}}{z_{a,b}}$$

Let $x_i = z_{i,b}$ and $y_j = z_{a,j}$. Then $z_{i,j} = \frac{1}{z_{a,b}} x_i y_j$. This shows that the point $[z_{i,j}]$ is uniquely determined by the vectors $(z_{0,b}, \ldots, z_{n,b})$ and $(z_{a,0}, \ldots, z_{a,m})$. The inverse map on the open set $U_{a,b} = \{z_{a,b} \neq 0\}$ is given by:

$$[z_{i,j}] \mapsto ([z_{0,b} : \cdots : z_{n,b}], [z_{a,0} : \cdots : z_{a,m}])$$

This map is regular and serves as a local inverse. Since this holds for any $z_{a,b} \neq 0$, the Segre map is an embedding. □

5.3 Open Immersion, Closed Immersion and Immersion

Intuition 13 Intuition behind Immersions of Schemes.

We need to analyze the relationships between schemes, such as embeddings and immersions, analogous to those in the theory of differentiable manifolds. Unlike the manifold case, however, the immersion of one scheme into another is more nuanced. This is largely because we define sheaves solely on open subsets of the scheme, without corresponding restrictions on closed subsets. Fortunately, commutative algebra provides a clear framework for understanding the ring-theoretic dynamics corresponding to these different types of immersions.

Let $(X, \mathscr{O}_X)$ and $(Z, \mathscr{O}_Z)$ be two schemes.

Definition 14 Immersions of Schemes.

A morphism $f : Z \to X$ is an open immersion if f is an open embedding of topological spaces, and the induced map of sheaves $f^\sharp : \mathscr{O}_X \to f_*\mathscr{O}_Z$ is an isomorphism.

A morphism of schemes $f : Z \to X$ is a closed immersion if f is a closed embedding of topological spaces, and the induced map of sheaves $f^\sharp : \mathscr{O}_X \to f_*\mathscr{O}_Z$ is surjective. The ideal sheaf $\ker\left(f^\sharp\right)$ is a quasi-coherent sheaf of ideals $\mathscr{I}$ of $\mathscr{O}_X$.

A morphism of schemes $f : Z \to X$ is an immersion if f can be decompose as $Z \to U \to X$, where $Z \to U$ is a closed immersion and $U \to X$ is an open immersion.

It is clear that when $f : Z \to X$ is an immersion, the image of Z under f is a locally closed subset of Z. This is equivalent to saying $f(Z)$ is open in its closure or it is the intersection of an open subset and a closed subset of X.

An important property of all kinds of immersion is they are all "locally", which means we can check them in the covering of X. To ensure this, we can consider the relation of stalks.

Proposition 15 Characterization of Immersions.

Let $\left(f, f^\sharp\right) : (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$ be a morphism.

1. $\left(f, f^\sharp\right)$ is an open immersion if and only if f induces a homeomorphism of Z with an open subset of X and $f_P^\sharp : \mathscr{O}_{X,f(P)} \to \mathscr{O}_{Z,P}$ is an isomorphism for every $P \in Z$.
2. $\left(f, f^\sharp\right)$ is an immersion if and only if f induces a homeomorphism of Z with a locally closed subset of X and $f_P^\sharp : \mathscr{O}_{X,f(P)} \to \mathscr{O}_{Z,P}$ is an epimorphism for every $P \in Z$.
3. Immersions are monomorphisms in the category of schemes. Moreover, the composite of immersions (resp. open immersions, resp. closed immersions) is an immersion (resp. an open immersion, resp. a closed immersion).

Proof.

(i) Open Immersion: The morphism $\left(f, f^\sharp\right)$ is an open immersion if and only if it is an isomorphism onto an open subscheme $(U, \mathscr{O}_X|_U)$, where $U = f(Z)$. This definition immediately implies that f is a homeomorphism onto an open subset $U \subseteq X$. Since isomorphism preserves the local structure, the induced stalk map $f_P^\sharp : \mathscr{O}_{X,f(P)} \to \mathscr{O}_{Z,P}$ must be an isomorphism for every $P \in Z$.

Conversely, if f is a homeomorphism onto an open set U and $f_P^\sharp$ is an isomorphism, the restricted sheaf morphism $\mathscr{O}_X|_U \to f_*\mathscr{O}_Z$ is an isomorphism because isomorphism is equivalent to isomorphism on stalks, thus $\left(f, f^\sharp\right)$ is an open immersion.

(ii) Immersion: An immersion factors as $Z \xrightarrow{i} U \xrightarrow{j} X$, where j is an open immersion and i is a closed immersion. The necessity is shown by factorization. Topologically, Z is a closed subset of U, which is open in X, making Z locally closed in X, and f is a homeomorphism onto this set. On the stalks, the map $f_P^\sharp$ is the composite of $j^\sharp$ (an isomorphism by (i)) and $i^\sharp$ (a surjection by the definition of closed immersion) , so $f_P^\sharp$ is an epimorphism (surjection) .

Conversely, assume f is a homeomorphism onto a locally closed set $W \subseteq X$ and $f_P^\sharp$ is an epimorphism. We can write W as a closed subset of some open $U \subseteq X$. Factoring f as $Z \xrightarrow{g} U \xrightarrow{j} X$, the stalk map $f_P^\sharp = g_P^\sharp \circ j_{f(P)}^\sharp$. Since $j^\sharp$ is an isomorphism and $f_P^\sharp$ is an epimorphism, $g_P^\sharp$ must also be an epimorphism. Since g is a homeomorphism onto a closed subset of U and its stalk maps are epimorphisms, g is a closed immersion. Thus $f = j \circ g$ is an immersion.

(iii) Monomorphisms and Composite: An immersion $f : Z \to X$ is a monomorphism. If $f \circ g = f \circ h$, the topological injectivity of f guarantees $g_{top} = h_{top}$.

On the sheaves, the stalk equality $g_y^\sharp \circ f_P^\sharp = h_y^\sharp \circ f_P^\sharp$ combined with the fact that $f_P^\sharp$ is an epimorphism (which is right-cancellable in the category of local rings) implies $g_y^\sharp = h_y^\sharp$, so $g = h$.

The composite of immersions is an immersion. This is because the composite of two locally closed maps is locally closed, and the composite of two epimorphisms on the stalks is an epimorphism. Specifically, the composite of open immersions is open (open in open is open, iso ∘ iso is iso), and the composite of closed immersions is closed (closed in closed is closed, sur ∘ sur is sur).

□

Here I state some useful examples about immersion:

Example 16 Closed Immersions.

Closed immersions come from "quotient".

1. **Point Inclusion in the Affine Line:**

   Let P be the origin in $\mathbb{A}_k^1$. The inclusion is given by the ring surjection:

   $$i : \operatorname{Spec}\left(k\left[x\right]/(x)\right) \hookrightarrow \operatorname{Spec}\left(k\left[x\right]\right) = \mathbb{A}_k^1$$

   The map on global sections is $k\left[x\right] \to k\left[x\right]/(x) \cong k$.

2. **Parabola in the Affine Plane:** The parabola defined by $y = x^2$ is a closed subscheme:

   $$i : \operatorname{Spec}\left(k\left[x, y\right]/(y - x^2)\right) \hookrightarrow \operatorname{Spec}\left(k\left[x, y\right]\right) = \mathbb{A}_k^2$$

3. **Non-Reduced Closed Subscheme:** A non-reduced closed subscheme (often called a "fat point") where the structure sheaf contains nilpotent elements:

   $$i : \operatorname{Spec}\left(k\left[x\right]/(x^2)\right) \hookrightarrow \operatorname{Spec}\left(k\left[x\right]\right) = \mathbb{A}_k^1$$

   The structure sheaf contains nilpotents represented by $x \pmod{x^2}$.

Example 17 Open Immersions.

Open immersions come from "localization".

4. **Complement of the Origin in the Affine Line:** This corresponds to the localization of the ring:

$$j : \mathrm{Spec}\left(k\left[x, x^{-1}\right]\right) \cong \mathrm{Spec}\left(k\left[x\right]_x\right) \hookrightarrow \mathrm{Spec}\left(k\left[x\right]\right) = \mathbb{A}_k^1$$

The corresponding topological space is $\mathbb{A}_k^1 \setminus \{0\}$.

5. **Standard Affine Patch of Projective Space:** The inclusion of the standard affine chart into projective n-space:

$$j : \mathbb{A}_k^n \hookrightarrow \mathbb{P}_k^n$$

This is realized by the open subset $D_+\left(x_0\right) = \mathrm{Proj}\left(k\left[x_0, \ldots, x_n\right]_{(x_0)}\right)$.

6. **General Open Affine Subscheme:** For any ring R and element $f \in R$, the principal open set is an open immersion:

$$j : \mathrm{Spec}\left(R_f\right) \hookrightarrow \mathrm{Spec}\left(R\right)$$

Example 18 Immersions.

A general immersion is a composition of a closed immersion followed by an open immersion ($Z \xrightarrow{\text{closed}} U \xrightarrow{\text{open}} X$) . Topologically, it is a homeomorphism onto a locally closed subset.

7. **A Closed Point in the Punctured Affine Line:** Let P_a be the point corresponding to $(x-a) \in k\left[x\right]$, where $a \neq 0$. The map factors through the punctured line $U = \mathbb{A}_k^1 \setminus \{0\}$:

$$f : \mathrm{Spec}\left(k\left[x\right]/(x-a)\right) \xrightarrow{\text{closed}} U \xrightarrow{\text{open}} \mathbb{A}_k^1$$

8. **Punctured Parabola in the Affine Plane:** Let $C \hookrightarrow \mathbb{A}_k^2$ be a closed immersion of the parabola defined by the ideal $(y-x^2) \subset k[x,y]$, and let P_0 be the origin $(0,0)$. The map $f : C \setminus \{P_0\} \longrightarrow \mathbb{A}_k^2$ is an immersion:

$$f : C \setminus \{P_0\} \xrightarrow{\text{open}} C \xrightarrow{\text{closed}} \mathbb{A}_k^2$$

$C \setminus \{P_0\}$ is an open subscheme of the closed subscheme C, hence it is a locally closed subscheme of $\mathbb{A}_k^2$.

Intuition 19 Intuition behind Closed Immersions into Affine Schemes.

We find the definition of a closed immersion, compared to that of an open immersion, is not intuitive. However, the examples provided exhibit strong regularity, namely that they are almost all induced by quotient homomorphisms. This leads us to hypothesize whether any closed immersion into an affine scheme is necessarily of this form.

Proposition 20 Closed Immersions into Affine Schemes.

Notations as above, we have:

1. The morphism of schemes $\phi : \mathrm{Spec}A/\mathfrak{a} \to \mathrm{Spec}A$ induced by the quotient homomorphism is a closed immersion.
2. Any closed immersion into an affine scheme is isomorphic to the above way.

Proof. (1) Let $\phi : A \to A/\mathfrak{a}$ be the quotient homomorphism. It induces a morphism of affine schemes $f : \mathrm{Spec}(A/\mathfrak{a}) \to \mathrm{Spec}(A)$, where a prime ideal $P \in \mathrm{Spec}(A/\mathfrak{a})$ is mapped to $f(P) = \phi^{-1}(P) \in \mathrm{Spec}(A)$.

1. Topological Properties: The image of f is given by:

$$f(\mathrm{Spec}(A/\mathfrak{a})) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \supseteq \ker(\phi) = \mathfrak{a}\} = V(\mathfrak{a})$$

For any closed subset $V(\bar{\mathfrak{b}}) \subseteq \mathrm{Spec}(A/\mathfrak{a})$ (where $\bar{\mathfrak{b}}$ is an ideal in $A/\mathfrak{a}$), let $\mathfrak{b} = \phi^{-1}(\bar{\mathfrak{b}})$. Its image is:

$$f(V(\bar{\mathfrak{b}})) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \supseteq \mathfrak{b}\} = V(\mathfrak{b})$$

This is a closed set in $\mathrm{Spec}(A)$. Since f is continuous and provides a bijection between prime ideals of $A/\mathfrak{a}$ and prime ideals of A containing $\mathfrak{a}$, f is a homeomorphism from $\mathrm{Spec}(A/\mathfrak{a})$ onto the closed subset $V(\mathfrak{a}) \subseteq \mathrm{Spec}(A)$.

2. Sheaf Condition: To show f is a closed immersion, we must check that the map of structure sheaves $f^\sharp : \mathscr{O}_{\mathrm{Spec}A} \to f_*\mathscr{O}_{\mathrm{Spec}(A/\mathfrak{a})}$ is surjective. It suffices to check this on the stalks. For any point $P \in \mathrm{Spec}(A/\mathfrak{a})$ and $\mathfrak{p} = f(P)$, the stalk map is:

$$f_P^\sharp : \mathscr{O}_{\mathrm{Spec}A,\mathfrak{p}} \to \mathscr{O}_{\mathrm{Spec}(A/\mathfrak{a}),P}$$

which corresponds to the map of local rings:

$$A_\mathfrak{p} \to (A/\mathfrak{a})_P \cong (A/\mathfrak{a})_{\mathfrak{p}/\mathfrak{a}}$$

Since the sequence $A \xrightarrow{\phi} A/\mathfrak{a} \to 0$ is exact, and localization is an exact functor, the localized sequence:

$$A_\mathfrak{p} \to (A/\mathfrak{a})_{\mathfrak{p}/\mathfrak{a}} \to 0$$

is also exact. This implies that the stalk map $f_P^\sharp$ is surjective for all P.

Therefore, f is a closed immersion.

(2) Let $f : Z \to \mathrm{Spec}\,A$ be a closed immersion. The proof proceeds in two stages: first establishing that Z is affine, and then constructing the algebraic isomorphism.

Step 1: The scheme Z is affine. Since f is a closed immersion, it is in particular a homeomorphism onto a closed subset of $\mathrm{Spec}\,A$. We identify the underlying topological space of Z with its image in $\mathrm{Spec}\,A$. Recall the criterion for affineness: a scheme S is affine if there exist elements $g_1, \dots, g_n \in \Gamma(S, \mathscr{O}_S)$ generating the unit ideal such that each open subset S_{g_i} (where g_i does not vanish) is affine.

For any point $P \in Z$, let $U_P \subset \mathrm{Spec}\,A$ be an affine open neighborhood such that $f^{-1}(U_P)$ is affine (this exists by the definition of a scheme/closed immersion). We can shrink U_P

to a basic open set $D(h_P) \subseteq U_P$ for some $h_P \in A$. The intersection $Z \cap D(h_P)$ is an open subscheme of the affine scheme $f^{-1}(U_P)$, and thus is affine.

Since $\operatorname{Spec} A$ is quasi-compact, we can cover Z (viewed as a closed subset) by finitely many such basic open sets $\{D(h_i)\}_{i=1}^m$. Note that we can extend this to a cover of the entire space $\operatorname{Spec} A$ by adding open sets in the complement of Z, but the focus is on Z. Let $\bar{h}_i$ denote the image of h_i under the map $f^\sharp : A \to \Gamma(Z, \mathscr{O}_Z)$. Since the $D(h_i)$ cover Z, the germs of $\bar{h}_i$ generate the structure sheaf everywhere, which implies that $\bar{h}_1, \dots, \bar{h}_m$ generate the unit ideal in the global section ring $\Gamma(Z, \mathscr{O}_Z)$. Furthermore, the open sets $Z_{\bar{h}_i} = f^{-1}(D(h_i))$ are precisely the affine schemes identified above. By the affineness criterion, Z is an affine scheme.

Step 2: Isomorphism with a quotient. Since Z is affine, let $Z \cong \operatorname{Spec} B$. The morphism $f : \operatorname{Spec} B \to \operatorname{Spec} A$ corresponds to a ring homomorphism $\phi : A \to B$. Let $\mathfrak{a} = \ker \phi$. By the First Isomorphism Theorem, the map ϕ factors as:

$$A \xrightarrow{\pi} A/\mathfrak{a} \xrightarrow{\bar{\phi}} B,$$

where π is the canonical projection and $\bar{\phi}$ is an injective ring homomorphism. This algebraic factorization induces a factorization of schemes:

$$\operatorname{Spec} B \xrightarrow{\theta} \operatorname{Spec}(A/\mathfrak{a}) \xrightarrow{g} \operatorname{Spec} A.$$

Here, g is the closed immersion corresponding to the surjection $A \to A/\mathfrak{a}$ (as proven in part (i)). Since the composition $f = g \circ \theta$ is a closed immersion by hypothesis, and g is a closed immersion, we claim θ must be an isomorphism.

Geometrically, the image of ϕ in B is isomorphic to $A/\mathfrak{a}$. Since f is a closed immersion, the map of sheaves $f^\sharp : \mathscr{O}_{\operatorname{Spec} A} \to f_*\mathscr{O}_{\operatorname{Spec} B}$ is surjective. This implies that the induced map on global sections $\phi : A \to B$ is surjective. Consequently, the injective map $\bar{\phi} : A/\mathfrak{a} \to B$ is also surjective, hence a ring isomorphism.

Alternatively, using the argument from the reference without assuming global surjectivity immediately: The map θ corresponds to the injection $\bar{\phi} : A/\mathfrak{a} \hookrightarrow B$. For any $h \in A/\mathfrak{a}$, the map on localizations $(A/\mathfrak{a})_h \to B_{\bar{\phi}(h)}$ is injective. Thus, the morphism of sheaves $\theta^\sharp : \mathscr{O}_{\operatorname{Spec}(A/\mathfrak{a})} \to \theta_*\mathscr{O}_{\operatorname{Spec} B}$ is injective. However, since f is a closed immersion, $f(Z)$ is homeomorphic to $V(\mathfrak{a})$, which is the underlying space of $\operatorname{Spec}(A/\mathfrak{a})$. Thus θ is a homeomorphism. Since it is a closed immersion (as a map between affines induced by a homomorphism), $\theta^\sharp$ is surjective. Being both injective and surjective, $\theta^\sharp$ is an isomorphism. Thus, $Z \cong \operatorname{Spec}(A/\mathfrak{a})$. □

Intuition 21 Intuition behind Closed Subschemes.

We wish to define the concept of a **closed subscheme** on a scheme X. This concept is not as straightforward as that of an **open subscheme**, which can be realized directly through a topological embedding and the restriction of the sheaf. Since we lack a direct definition for the structure sheaf restricted to a closed set, we are forced to define this concept via the notion of a **closed immersion**.

A closed subscheme of X is defined as an equivalence class of closed immersions $\iota : Z \to X$. Two closed immersions $\iota_1 : Z_1 \to X$ and $\iota_2 : Z_2 \to X$ are said to be isomorphic (i.e., they define the same closed subscheme) if there exists an isomorphism $\alpha : Z_1 \xrightarrow{\cong} Z_2$ such that the following diagram commutes:

$$\begin{tikzcd}
Z_1 \arrow[r, "\iota_1"] \arrow[d, "\alpha", "\cong"'] & X \\
Z_2 \arrow[ru, "\iota_2"'] &
\end{tikzcd}$$

This confirms that a closed subscheme is precisely the **isomorphism class** of closed immersions into X.

Among these isomorphism classes, we often wish to select the most canonical or simplest representative. Specifically, we aim to choose a representative Z that is a **reduced scheme**, which we then call the **induced reduced closed subscheme**.

Proposition 22 Reduced Induced Closed Subscheme.

For any closed subset Y of a scheme X there is a unique reduced closed subscheme $(Y, \mathcal{O}_Y)$ of X whose topological space is Y, such that for any closed immersion $(Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ where the image of Z in X containing Y, there's a unique morphism of schemes $(Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$, satisfies $((Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)) = ((Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)) \circ ((Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X))$. We call this subscheme reduced induced closed subscheme.

Proof. The proof proceeds by first establishing the result for affine schemes using commutative algebra, and then gluing the local results to cover the general case.

Step 1: The Affine Case. Suppose $X = \operatorname{Spec} A$ is an affine scheme. Let $Y \subseteq X$ be a closed subset. Ideally, Y corresponds to a radical ideal $\mathfrak{a} \subseteq A$ such that $Y = V(\mathfrak{a})$ and $\mathfrak{a} = \sqrt{\mathfrak{a}}$. We define the reduced induced subscheme as $(Y, \mathcal{O}_Y) := \operatorname{Spec}(A/\mathfrak{a})$. Since $A/\mathfrak{a}$ is a reduced ring (has no non-zero nilpotents), this scheme is reduced. The canonical projection $\pi : A \to A/\mathfrak{a}$ induces a closed immersion $\operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec} A$ with image $V(\mathfrak{a}) = Y$.

Proof of Universal Property (Affine): Let $f : Z \to X$ be any closed immersion such that $f(Z) \supseteq Y$. Since X is affine, by **Proposition** 20, Z is affine and isomorphic to $\operatorname{Spec}(A/\mathfrak{b})$ for some ideal $\mathfrak{b} \subseteq A$. The closed immersion f corresponds to the projection $A \to A/\mathfrak{b}$. The condition $f(Z) \supseteq Y$ translates algebraically to $V(\mathfrak{b}) \supseteq V(\mathfrak{a})$. By the properties of Zariski topology, this inclusion implies:

$$\sqrt{\mathfrak{b}} \subseteq \sqrt{\mathfrak{a}}.$$

Since we constructed $\mathcal{O}_Y$ using the radical ideal $\mathfrak{a} = \sqrt{\mathfrak{a}}$, we have the chain of inclusions:

$$\mathfrak{b} \subseteq \sqrt{\mathfrak{b}} \subseteq \sqrt{\mathfrak{a}} = \mathfrak{a}.$$

The inclusion $\mathfrak{b} \subseteq \mathfrak{a}$ guarantees the existence and uniqueness of a ring homomorphism $A/\mathfrak{b} \to A/\mathfrak{a}$ making the diagram of rings commute (factoring the projection $A \to A/\mathfrak{a}$). Taking spectra, this yields a unique morphism of schemes $g : \operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec}(A/\mathfrak{b})$ satisfying the commutative diagram in the proposition.

Step 2: Uniqueness of the Morphism (Global Gluing). Now let X be an arbitrary scheme. Suppose the reduced induced subscheme $(Y, \mathcal{O}_Y)$ exists. We show the morphism g in the universal property is unique. Cover X by affine open subsets $\{U_i\}_{i \in I}$. Let $Y_i = Y \cap U_i$ and $Z_i = f^{-1}(U_i)$. The restriction of the problem to each U_i gives a diagram of schemes over the affine U_i. By the affine case proven in Step 1, for each i, there exists a unique morphism $g_i : Y_i \to Z_i$ making the local diagram commute. On the intersection $U_i \cap U_j$, the uniqueness implies that $g_i|_{Y \cap U_i \cap U_j} = g_j|_{Y \cap U_i \cap U_j}$. Thus, by the sheaf property of morphisms, these local morphisms $\{g_i\}$ glue to a unique global morphism $g : Y \to Z$.

Step 3: Existence of the Scheme (Global Construction). We construct $(Y, \mathscr{O}_Y)$ by gluing. Cover X by affine open sets $U_i = \operatorname{Spec} A_i$. For each i, let $Y_i = Y \cap U_i$. Let $\mathfrak{a}_i \subseteq A_i$ be the ideal defining Y_i (specifically, $\mathfrak{a}_i = \bigcap_{\mathfrak{p} \in Y_i} \mathfrak{p}$). We define the local scheme $(Y_i, \mathscr{O}_{Y_i}) := \operatorname{Spec}(A_i/\mathfrak{a}_i)$.

To glue these into a global scheme, consider the intersection $U_i \cap U_j$. Note that $U_i \cap U_j$ is open in U_i, so it can be covered by basic open affine sets $U_{ijk} \subseteq U_i \cap U_j$. The induced structure on $Y \cap U_{ijk}$ coming from Y_i is simply the localization of $A_i/\mathfrak{a}_i$, which is naturally isomorphic to the induced structure coming from Y_j (since both are the unique reduced structures on the same affine set). Therefore, we have canonical isomorphisms $\theta_{ij} : (Y_i, \mathscr{O}_{Y_i})|_{U_i \cap U_j} \xrightarrow{\sim} (Y_j, \mathscr{O}_{Y_j})|_{U_i \cap U_j}$ satisfying the cocycle condition. We glue these affine schemes $(Y_i, \mathscr{O}_{Y_i})$ along these isomorphisms to obtain a global scheme $(Y, \mathscr{O}_Y)$. Since being reduced is a local property, and each affine piece is reduced, the global scheme is reduced. The inclusion morphisms $Y_i \hookrightarrow U_i$ glue to a global closed immersion $Y \hookrightarrow X$.

This constructs the required scheme, and Step 2 guarantees it satisfies the universal property. □

When X is an affine scheme $\operatorname{Spec} A$, we know that any closed immersion is isomorphic with $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$, and $Y = V(\mathfrak{a})$. Since $(Y, \mathscr{O}_Y)$ is reduced, the ideal $\mathfrak{a}$ must satisfies that $\sqrt{\mathfrak{a}} = \mathfrak{a}$. Such kind of the ideal $\mathfrak{a}$ is uniquely determined by $Y = V(\mathfrak{a})$.

5.4 Fiber Product and Base Change

Intuition 23 Intuition behind Fiber Product and Base Change.

Next, we will introduce the extremely important concepts in algebraic geometry, namely the fiber product and base change; in fact, these two concepts are fundamentally equivalent.To do this, we need some preparation.

First, we have become familiar with the category of schemes (Sch) and the category of affine schemes (**AffSch**). We will construct a related class of categories, specifically the category of S-schemes (**Sch**/S). The fiber product is then realized as the product object (or more accurately, the pullback/fiber product) within this category **Sch**/S.

Definition 24 the Category of S-schemes.

The category of S-schemes, denoted Sch/S, is defined as follows:

i. **Objects Ob(Sch/S):** An object X is a scheme X together with a structural morphism $X \to S$.

ii. **Morphisms Mor(Sch/S):** A morphism $f : X \to Y$ is a scheme morphism such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\
& S &
\end{tikzcd}$$

iii. **Composition:** Composition $f \circ g$ is defined as usual in the category of schemes.

Definition 25 Fiber Product and Base Change.

The **fiber product** is the product of two S-schemes X and Y in the category Sch/S. It is the unique scheme $X \times_S Y$ such that the square commutes, satisfying the following universal property:

The scheme $X \times_S Y$ together with the projection S-morphisms $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ makes the following square commutative:

$$\begin{tikzcd}
X \times_S Y \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & S
\end{tikzcd}$$

Moreover, for any S-scheme T and any S-morphisms $\pi : T \to X$ and $\rho : T \to Y$, there exists a unique S-morphism $h : T \to X \times_S Y$ such that the larger diagram commutes:

$$\begin{tikzcd}
T \arrow[dr, dashed, "h"] \arrow[drr, bend left, "\rho"] \arrow[ddr, bend right, "\pi"'] & & \\
& X \times_S Y \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & S
\end{tikzcd}$$

This is equivalent to satisfying $p \circ h = \pi$ and $q \circ h = \rho$. The scheme $X \times_S Y$ is unique up to unique isomorphism.

We also say that the S-morphism q is the **base change** of f along with g, and p is the base change of g along with f

We need to guarantee the existence and uniqueness of the fiber product:

1. **The Uniqueness**

   The uniqueness of the fibred product (up to unique isomorphism) is an immediate consequence of the **universal property** of the fiber product in category theory.

2. **Base Case: Affine Schemes** (The Foundation)

   If all three schemes are affine, $X = \mathrm{Spec}A, Y = \mathrm{Spec}B, S = \mathrm{Spec}C$, the fibre product exists and is defined algebraically by the tensor product over the base ring:

$$X \times_S Y = \mathrm{Spec}(A \otimes_C B)$$

3. **Locality Check**

   The fibre product commutes with taking open subschemes: $p^{-1}(U)$ is the fibre product of U and Y over S.

4. **Gluing Principle (Step 3)**

   If a scheme X can be covered by open subschemes U_i such that all local products $U_i \times_S Y$ exist, then the global product $X \times_S Y$ exists. This is achieved by checking that the local products are consistent on the overlaps $(U_i \cap U_j) \times_S Y$ and then applying the scheme gluing theorem.

5. **Affine Base, One Affine Factor (S = Affine, Y = Affine)**

   We combine the affine base case (Step 1) with the gluing principle (Step 3). By covering X with affine open subschemes and applying Step 1 and Step 3, we show the product exists when S and Y are affine.

6. **Affine Base, Arbitrary Factors (S = Affine)**

   The fibre product is symmetric. By exchanging X and Y and applying Step 4 and Step 3 again, the product exists for any arbitrary schemes X and Y over an affine base S.

7. **General Case (Arbitrary Base S)**

   The final step is to cover the base scheme S with affine open subschemes S_i. This reduces the problem to a family of problems over affine bases S_i (Step 5). The final global product $X \times_S Y$ is obtained by gluing these local products $X_i \times_{S_i} Y_i$ together using the gluing principle (Step 3).

There are some useful properties of fiber product:

Proposition 26 Properties of Fiber Product.

Let S be a scheme. For any S-schemes X, Y, Z and any S-scheme S', the following natural isomorphisms hold:

1. **Symmetry (Commutativity):** The order of the factors does not matter.

$$X \times_S Y \cong Y \times_S X \tag{5.1}$$

2. **Associativity:** The grouping of factors does not matter for multiple fiber products over the same base S.

$$(X \times_S Y) \times_S Z \cong X \times_S (Y \times_S Z) \tag{5.2}$$

3. **Identity Base Change:** The fiber product with the base scheme itself, where the map is the identity, is the scheme itself.

$$X \times_S S \cong X \tag{5.3}$$

4. **Identity Fiber Product:** If the structure morphism $X \to S$ is the identity id_X, the product is X.

$$X \times_X X \cong X \tag{5.4}$$

5. **Locality (Commutativity of Fiber Product and Base Change):**

   If S' is a scheme over S (via a structure morphism $h : S' \to S$), the operation of taking the fiber product $X \times_S Y$ and then pulling it back to the new base S' (i.e., base changing) is the same as first pulling back X and Y separately to S', and then taking their fiber product over S':

   $$(X \times_S Y) \times_S S' \cong (X \times_S S') \times_{S'} (Y \times_S S') \tag{5.5}$$

   This property is also crucial in the existence proof of the general fiber product.

As for the proof, they are just the following commutative graph:

$$\begin{tikzcd}
T \arrow[dr, dashed, "\pi=\rho"] \arrow[drr, bend left, "\rho"] \arrow[ddr, bend right, "\pi"'] & & \\
& X \arrow[r, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] & X \arrow[d, "\mathrm{id}"'] \\
& X \arrow[r, "\mathrm{id}"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[dr, dashed, "(\rho,\pi)"] \arrow[drr, bend left, "\rho"] \arrow[ddr, bend right, "\pi"'] & & \\
& X \times_S Y \arrow[r, "p"'] \arrow[d, "q"] & Y \arrow[d, "g"'] \\
& X \arrow[r, "f"] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[dr, dashed, "\pi"] \arrow[drr, bend left, "\rho"] \arrow[ddr, bend right, "\pi"'] & & \\
& X \arrow[r, "f"'] \arrow[d, "\mathrm{id}"] & S \arrow[d, "\mathrm{id}"'] \\
& X \arrow[r, "f"] & S
\end{tikzcd}$$

$$\begin{tikzcd}
& & & & T \arrow[dllll, dashed, "(p_1\circ\pi,(q_1\circ\pi,\rho))"'] \arrow[ddlll, "\pi"] \arrow[ddd, bend left, "\rho"] \\
X \times_S (Y \times_S Z) \arrow[dr, "(p',q_2\circ q')"'] \arrow[drrr, "q'"] \arrow[ddr, bend right, "p'"'] & & & & \\
& X \times_S Y \arrow[r, "q_1"'] \arrow[d, "p_1"] & Y \arrow[d, "g"'] & Y \times_S Z \arrow[l, "q_2"'] \arrow[d, "p_2"'] & \\
& X \arrow[r, "f"] & S & Z \arrow[l, "h"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& T \arrow[dl, "\pi"'] \arrow[dr, "\rho"] \arrow[ddrr, dashed, bend left, "((p_1\circ\pi,p_2\circ\rho),q_1\circ\pi)"] & & \\
X \times_S S' \arrow[r, "q_1"] \arrow[dd, "p_1"] & S' \arrow[dd, "h"] & Y \times_S S' \arrow[l, "q_2"'] \arrow[dd, "p_2"] & \\
& & & (X \times_S Y) \times_S S' \arrow[ull, bend left, "q''"'] \arrow[ddll, bend left, "p''"] \\
X \arrow[r, "f"] & S & Y \arrow[l, "g"] & \\
& X \times_S Y \arrow[ul, bend left, "p'"] \arrow[ur, bend right, "q'"'] & &
\end{tikzcd}$$

To provide a more intuitive grasp of fiber products (and to facilitate their explicit computation), we introduce the following proposition. The reader will observe that the fiber product is, in effect, a generalization of 'intersections' and 'preimages' from classical geometry. Beyond merely recovering these sets at the topological level, the fiber product endows the resulting space with an 'appropriate' scheme structure (sheaf of rings)

Proposition 27 Fiber Product in Sets.

Let $f : X \to S$ and $g : Y \to S$ be morphisms of sets. The fiber product of X and Y over S, denoted by $X \times_S Y$, is defined as the set $\{(x, y) \in X \times Y \mid f(x) = g(y)\}$. The following properties hold:

1. **Direct Product:** If S is a point, then $X \times_S Y \cong X \times Y$.
2. **Intersection:** If $X, Y \subset S$ and f, g are inclusion maps, then $X \times_S Y \cong X \cap Y$.
3. **Preimage:** If $Y \subset S$ and g is the inclusion map, then $X \times_S Y \cong f^{-1}(Y)$, where the isomorphism is given by the projection onto X.
4. **Equalizer:** If $X = Y$, the fiber product identifies the subset of X where the two maps f and g agree, i.e., $X \times_{S \times S} S \cong \{x \in X \mid f(x) = g(x)\}$.

Proof. We prove each statement based on the set-theoretic definition of the fiber product:

1. If $S = \{s\}$, the condition $f(x) = g(y)$ is satisfied for all $x \in X$ and $y \in Y$ since $f(x) = s$ and $g(y) = s$ are constant. Thus, $X \times_S Y = X \times Y$.
2. For inclusions $X \hookrightarrow S$ and $Y \hookrightarrow S$, the condition $f(x) = g(y)$ implies $x = y$ in S. Since $x \in X$ and $y \in Y$, it follows that $x \in X \cap Y$.
3. If g is an inclusion, the condition $f(x) = g(y)$ becomes $f(x) = y$. Since y must be an element of Y, this is equivalent to $f(x) \in Y$. The first projection then maps $(x, f(x))$ to $x \in f^{-1}(Y)$.
4. Let $\phi, \psi : X \to S$ be two maps. Considering the fiber product of $X \xrightarrow{(\phi,\psi)} S \times S$ and $S \xrightarrow{\Delta} S \times S$, the condition $(\phi(x), \psi(x)) = \Delta(s) = (s, s)$ implies $\phi(x) = \psi(x) = s$. This yields the equalizer $\{x \in X \mid \phi(x) = \psi(x)\}$.

□

Intuition 28 Intuition behind Fiber as a Fiber Product.

One of the most crucial roles of the fiber product ($\times_Y$) in scheme theory is to represent the fiber of a scheme morphism.

Given a morphism of schemes $f : X \to Y$, we are interested in the preimage of a specific point y in the target space Y, i.e., the set of points $f^{-1}(y)$. In topology, one can indeed obtain these points by simply taking the preimage in the topological space, $f_{\text{top}}^{-1}(y)$. However, this purely point-set operation fails to inherit the sheaf structure, preventing us from obtaining an object with a proper scheme structure.

To construct a geometric object that correctly preserves the scheme structure (i.e., the scheme corresponding to $f^{-1}(y)$), we take the fiber product of the source scheme X with the spectrum of the residue field Spec $k(y)$ associated with the point y:

$$X_y := X \times_Y \operatorname{Spec} k(y)$$

Through this construction, it can be proven that the underlying topological space $|X_y|$ of the resulting fiber scheme is homeomorphic to the preimage $f_{\text{top}}^{-1}(y)$ in the topological sense. Thus, the fiber product not only yields the correct set of points but also endows them with a natural local ring and algebraic structure.

Proposition 29 Fiber as a Fiber Product.

Let $f : X \to Y$ be a morphism of schemes, y a point in Y, and $k(y)$ the residue field of $\mathcal{O}_{Y,y}$. The projection $X \times_Y \operatorname{Spec} k(y) \to X$ induces a homeomorphism onto the underlying topological space of $f^{-1}(y)$.

Proof. The problem is local on Y and X. We may assume that $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ are affine, and f is induced by a ring homomorphism $\phi : A \to B$. The point $y \in Y$ corresponds to a prime ideal $\mathfrak{p} \subset A$.

Part 1: The Algebraic Structure of the Fiber

The fiber $X \times_Y \operatorname{Spec} k(y)$ corresponds to the ring $R = B \otimes_A k(y)$. The residue field $k(y)$ is defined as the quotient of the local ring $\mathcal{O}_{Y,y} \cong A_\mathfrak{p}$ by its maximal ideal $\mathfrak{p}A_\mathfrak{p}$:

$$k(y) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$$

Substituting this into the definition of R:

$$R \cong B \otimes_A (A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})$$

Since $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \cong (A \otimes_A A_\mathfrak{p})/(\mathfrak{p} \otimes_A A_\mathfrak{p})$, we apply the property that tensoring commutes with quotienting:

$$R \cong (B \otimes_A A_\mathfrak{p})/(\mathfrak{p}A_\mathfrak{p} \otimes_A B)$$

We use the standard notation $B_\mathfrak{p} = B \otimes_A A_\mathfrak{p}$ (the localization of B with respect to $\phi(A \setminus \mathfrak{p})$) and recognize the ideal in the denominator as $\mathfrak{p}B_\mathfrak{p}$.

$$R \cong B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$$

Thus, the fiber is:

$$X \times_Y \operatorname{Spec} k(y) \cong \operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$$

Part 2: Homeomorphism on Topological Spaces (Point Correspondence)

We establish a bijection between the points of $|X \times_Y \operatorname{Spec} k(y)|$ and $f^{-1}(y)$. A prime ideal $\mathfrak{q}'$ of $R = B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ corresponds to a prime ideal $\mathfrak{q}_{\text{loc}} \subset B_\mathfrak{p}$ such that $\mathfrak{q}_{\text{loc}} \supseteq \mathfrak{p}B_\mathfrak{p}$.

The points of $B_\mathfrak{p}$ correspond to prime ideals $\mathfrak{q} \subset B$ that are disjoint from the image of the localization set, $\phi(A \setminus \mathfrak{p})$.

$$\mathfrak{q} \cap \phi(A \setminus \mathfrak{p}) = \emptyset \quad \Longleftrightarrow \quad \phi^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}$$

The containment condition $\mathfrak{q}_{\text{loc}} \supseteq \mathfrak{p}B_\mathfrak{p}$ translates to:

$$\mathfrak{q} \supseteq \phi(\mathfrak{p})$$

Taking the preimage under ϕ:

$$\phi^{-1}(\mathfrak{q}) \supseteq \phi^{-1}(\phi(\mathfrak{p})) \supseteq \mathfrak{p}$$

Combining the two conditions ($\phi^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}$ and $\phi^{-1}(\mathfrak{q}) \supseteq \mathfrak{p}$), we conclude:

$$\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$$

This is precisely the algebraic condition for the geometric statement $f(\mathfrak{q}) = y$, which means $\mathfrak{q} \in f^{-1}(y)$. Thus, the topological spaces are in bijection.

Part 3: Factorization as an Embedding

The projection $\pi : X \times_Y \operatorname{Spec} k(y) \to X$ is induced by the composite homomorphism $B \to B_\mathfrak{p} \to B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$. The projection factors as:

$$\begin{tikzcd}
\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) \arrow[r, "g"] \arrow[rr, bend left, "\pi"] & \operatorname{Spec}(B_\mathfrak{p}) \arrow[r, "h"] & \operatorname{Spec} B = X
\end{tikzcd}$$

i. h: Open Immersion.textbf The map h is induced by the localization map $B \to B_\mathfrak{p}$. The image of h is the principal open subset $D(\phi(A \setminus \mathfrak{p}))$, hence h is an open immersion.

ii. g: Closed Immersion.textbf The map g is induced by the surjective map $B_\mathfrak{p} \to B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$. By definition, this makes g a closed immersion.

Since a closed immersion followed by an open immersion is a local closed immersion (an embedding on the underlying topological space), the map π induces a homeomorphism onto its image, which is $f^{-1}(y)$.

□

Two other crucial concepts derived from the fiber product are the **diagonal morphism** and the **graph morphism**. These two morphisms are simply defined by the pairs of universal arrows, (id, id) and (id, f), respectively, and they are defined by the requirement that the following diagrams commute. These constructions are fundamental in defining properties of schemes, such as separatedness (via the diagonal morphism).

Definition 30 Diagonal Morphism.

Let X be an S-scheme, with structure morphism $p : X \to S$. The **diagonal morphism** $\Delta_X : X \to X \times_S X$ is the unique morphism induced by the pair of identity maps $(\mathrm{id}_X, \mathrm{id}_X)$.

It makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[drr, bend left, "\mathrm{id}"] \arrow[dr, dashed, "\Delta_{X/Y}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& X \times_Y X \arrow[r, "q"'] \arrow[d, "p"] & X \arrow[d, "f"'] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

Here, π_1 and π_2 are the projection maps of the fiber product.

Definition 31 Graph Morphism.

Let $f : X \longrightarrow Y$ be a morphism of S-schemes. The **graph morphism** $\Gamma_f : X \longrightarrow X \times_S Y$ is the unique morphism induced by the pair (id_X, f). It makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[drr, bend left, "f"] \arrow[dr, dashed, "\Gamma_f"'] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& X \times_S Y \arrow[r, "q"'] \arrow[d, "p"] & Y \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

Here, π_1 and π_2 are the projection maps of the fiber product, and q is the structure morphism $Y \longrightarrow S$.

5.5 Integral Schemes and Divisors

Intuition 32 Intuition behind Integral Schemes.

In this section, we discuss one of the most fundamental classes of objects in scheme theory: integral schemes. Their excellent algebraic properties, combined with the powerful theory of divisors, carry immense significance for both number theory and algebraic geometry. This framework provides a rigorous foundation for many geometric intuitions, such as the fact that rational functions possess an equal number of zeros and poles (counted with multiplicity). Furthermore, divisor theory serves as the cornerstone for the study of algebraic curves and the Riemann-Roch theorem, both of which are indispensable for engaging with 'true' geometry.

We begin by introducing a relatively weaker property of schemes known as being reduced. A scheme is said to be reduced if its sections over any open set contain no non-zero nilpotent elements. While perhaps less restrictive than integrality, this property plays a fundamental role in characterizing the local algebraic structure of schemes.

Definition 33 Reduced Scheme.

A scheme $(X, \mathscr{O}_X)$ is called **reduced** if for every open subset $U \subseteq X$, the ring $\mathscr{O}_X(U)$ has no non-zero nilpotent elements (i.e., it is a reduced ring).

A key advantage of being reduced is that it is a local property, meaning it can be checked stalkwise.

Proposition 34 Local Characterization of Reducedness.

A scheme $(X, \mathscr{O}_X)$ is reduced if and only if for every point $x \in X$, the stalk $\mathscr{O}_{X,x}$ is a reduced ring.

Proof. ($\Rightarrow$) Suppose X is reduced. Let $x \in X$ and let $s_x \in \mathscr{O}_{X,x}$ be a germ such that $(s_x)^n = 0$ for some $n > 0$. We represent s_x by a section $s \in \mathscr{O}_X(U)$ on some open neighborhood U of x. The equation $(s_x)^n = 0$ in the stalk implies that $(s^n)_x = 0$. By the definition of the stalk, there exists a smaller neighborhood $V \subseteq U$ of x such that $s^n|_V = 0$ in $\mathscr{O}_X(V)$. Since $\mathscr{O}_X(V)$ is a reduced ring by hypothesis, $s^n|_V = 0$ implies $s|_V = 0$. Consequently, the germ $s_x = (s|_V)_x$ is zero. Thus $\mathscr{O}_{X,x}$ has no nilpotents.

($\Leftarrow$) Suppose all stalks $\mathscr{O}_{X,x}$ are reduced. Let $U \subseteq X$ be open and let $s \in \mathscr{O}_X(U)$ be a nilpotent element, say $s^n = 0$. For any point $x \in U$, the germ satisfying $(s_x)^n = (s^n)_x = 0$. Since $\mathscr{O}_{X,x}$ is reduced, we must have $s_x = 0$. Since a section of a sheaf is uniquely determined by its germs (being the zero section is a local property), $s_x = 0$ for all $x \in U$ implies $s = 0$. Thus $\mathscr{O}_X(U)$ is reduced. □

We now formally define integral schemes and locally integral schemes: a scheme is locally integral if its rings of sections (or equivalently, its stalks) are all integral domains.

Definition 35 Integral and Locally Integral Schemes.

(i) A scheme $(X, \mathscr{O}_X)$ is called **integral** if for every non-empty open subset $U \subseteq X$, the ring $\mathscr{O}_X(U)$ is an integral domain.

(ii) A scheme $(X, \mathscr{O}_X)$ is called **locally integral** if for every point $x \in X$, the stalk $\mathscr{O}_{X,x}$ is an integral domain.

The most classical characterization, or criterion, for an integral scheme is that integrality is equivalent to the combination of being irreducible and reduced.

Proposition 36 Characterization of Integral Schemes.

A scheme X is integral if and only if it is both **reduced** and **irreducible** as a topological space.

Proof. ($\Rightarrow$) Suppose X is integral. First, since $\mathscr{O}_X(U)$ is an integral domain, it is reduced. Thus X is reduced. Second, suppose X were reducible. Then X can be written as the union of two proper closed subsets, which implies there exist two disjoint non-empty open subsets $U_1, U_2 \subset X$. Let $U = U_1 \cup U_2$. The structure sheaf on disjoint unions is a product: $\mathscr{O}_X(U) \cong \mathscr{O}_X(U_1) \times \mathscr{O}_X(U_2)$. Consider the elements $e_1 = (1, 0)$ and $e_2 = (0, 1)$ in this product ring. Neither is zero, but their product is zero. Thus $\mathscr{O}_X(U)$ has zero divisors, contradicting the assumption that X is integral. Hence X must be irreducible.

($\Leftarrow$) Suppose X is reduced and irreducible. Let U be a non-empty open set. Since X is reduced, $\mathscr{O}_X(U)$ is a reduced ring. We must show it has no zero divisors. Let $f, g \in \mathscr{O}_X(U)$ such that $fg = 0$. We want to show $f = 0$ or $g = 0$. Define the open sets of non-vanishing germs:

$$U_f = \{x \in U \mid f_x \neq 0\} \quad \text{and} \quad U_g = \{x \in U \mid g_x \neq 0\}.$$

(Note: On a reduced scheme, $f_x \neq 0$ is equivalent to f not vanishing on any neighborhood of x). Since stalks are localizations of global sections, the relation $fg = 0$ implies $f_x g_x = 0$ in $\mathscr{O}_{X,x}$. Since $\mathscr{O}_{X,x}$ is reduced (by the previous Proposition), if $f_x g_x = 0$, then every minimal prime contains $f_x g_x$. However, this local argument is subtle. Let's proceed topologically. Since X is reduced, $f = 0$ if and only if $U_f = \emptyset$. For any x, $f_x g_x = 0$. If $x \in U_f$, then $f_x \neq 0$. In a reduced local ring, zero divisors are unions of minimal primes... actually, there is a simpler argument using the property of irreducibility directly.

Since $fg = 0$, for any x, $f_x g_x = 0$. It can be shown that $U_f \cap U_g = \emptyset$. (If $x \in U_f \cap U_g$, then $f_x \neq 0$ and $g_x \neq 0$. In a reduced ring, this doesn't immediately imply contradiction unless it's a domain, but we are proving it's a domain).

Refined Argument: Let's use the sheaf property. Suppose $f \neq 0$ and $g \neq 0$. Since X is reduced, there exist non-empty open sets $V_f \subseteq U$ where $f|_{V_f}$ is nowhere zero (invertible if we shrink enough? No, just non-zero germs) and $V_g \subseteq U$ where $g|_{V_g}$ is non-zero. Actually, the standard argument is: Let $Z(f)$ and $Z(g)$ be the closed sets where the sections vanish. Since $fg = 0$, $X = Z(f) \cup Z(g)$. Since X is irreducible, one of them must be the whole space X. If $Z(f) = X$, then f is zero everywhere locally. Since X is reduced, $f = 0$. If $Z(g) = X$, then $g = 0$. Thus $\mathscr{O}_X(U)$ is an integral domain. □

Correspondingly, we can see that being locally integral is equivalent to a weaker condition: a scheme is locally integral if and only if it is reduced and its irreducible components are disjoint. From this, it is evident that an integral scheme is necessarily locally integral, as the irreducibility of the former vacuously satisfies the disjointness of components.

Proposition 37 Locally Integral Schemes (Noetherian Case).

Let X be a Noetherian scheme (meaning the underlying topological space is Noetherian). Then X is **locally integral** if and only if X is **reduced** and its irreducible components are **disjoint**. Under these conditions, the irreducible components coincide with the connected components.

Proof. First, recall that a Noetherian space X has finitely many irreducible components $X = X_1 \cup \cdots \cup X_n$.

($\Leftarrow$) Suppose X is reduced and the components X_i are disjoint. Since there are finitely many components, disjointness implies each X_i is both open and closed (clopen). Thus $X \cong \bigsqcup X_i$ as schemes. For any point $x \in X$, x belongs to exactly one component, say X_k. Since X_k is open, $\mathscr{O}_{X,x} \cong \mathscr{O}_{X_k,x}$. The scheme X_k is reduced (inherited from X) and irreducible. By the previous proposition, X_k is an integral scheme. Thus its stalk $\mathscr{O}_{X_k,x}$ is an integral domain. Hence X is locally integral.

($\Rightarrow$) Suppose X is locally integral. This means for every x, $\mathscr{O}_{X,x}$ is an integral domain.

1. **Reducedness:** Since integral domains are reduced, all stalks are reduced. By **Proposition 34**, X is reduced.

2. **Disjointness:** Suppose for contradiction that two distinct irreducible components X_1 and X_2 intersect. Let $x \in X_1 \cap X_2$. Recall that the irreducible components of a scheme passing through x correspond bijectively to the **minimal prime ideals** of the local ring $\mathcal{O}_{X,x}$. Specifically, if $\mathfrak{p} \subset \mathcal{O}_{X,x}$ is a minimal prime, $V(\mathfrak{p})$ inside $\operatorname{Spec} \mathcal{O}_{X,x}$ corresponds to the germ of an irreducible component at x. Since x lies in both X_1 and X_2, the ring $\mathcal{O}_{X,x}$ must have at least two distinct minimal primes corresponding to these geometric branches. However, $\mathcal{O}_{X,x}$ is an integral domain by hypothesis. An integral domain has exactly one minimal prime ideal: the zero ideal (0). This forces the branches to coincide locally, which implies $X_1 = X_2$ (since irreducible components are determined by their generic points, or by closure of local structure). This is a contradiction.

Therefore, the irreducible components must be disjoint. □

The most distinctive hallmark of an integral scheme is the existence of a generic point: a unique, dense single-point subset whose closure constitutes the entire scheme.

Proposition 38 Generic Point of an Integral Scheme.

Let X be an integral scheme. Then there exists a unique point $\xi \in X$, called the **generic point**, such that the closure of $\{\xi\}$ is the entire space X (i.e., $\overline{\{\xi\}} = X$).

Proof. By definition, an integral scheme X is irreducible. In a spectral space, irreducibility implies the existence of a unique generic point.

- **Existence:** Let $U = \operatorname{Spec} A$ be any non-empty affine open subset. Since X is integral, A is an integral domain. The zero ideal $(0) \subset A$ is a prime ideal, corresponding to a point $\xi \in U$. The closure of ξ in U is $V((0)) = \operatorname{Spec} A = U$. Since U is open and non-empty in an irreducible space X, U is dense in X. Thus $\overline{\{\xi\}} = \bar{U} = X$.

- **Uniqueness:** If η is another generic point, then $\eta \in \overline{\{\xi\}}$ implies $\xi \in$ open neighborhood of η. In a T_0-space (which all schemes are), generic points of the same irreducible component must coincide.

□

Due to the privileged nature of this point, its local ring is referred to as the function field of the integral scheme. From the perspective of commutative algebra, for an affine integral scheme $\operatorname{Spec} A$ where A is an integral domain, the function field is simply the field of fractions of A, obtained by localizing at the zero ideal (0).

Definition 39 Function Field.

Let X be an integral scheme with generic point ξ. The stalk of the structure sheaf at the generic point, $\mathcal{O}_{X,\xi}$, is a field. This field is called the **function field** of X and is denoted by $K(X)$.

Explicitly, if $U = \operatorname{Spec} A$ is any non-empty open affine subset, then $K(X)$ is isomorphic to the fraction field of A, denoted $\operatorname{Frac}(A)$. The elements of $K(X)$ are called **rational functions** on X.

Proposition 40 Injectivity of Restriction and Stalk Maps.

Let X be an integral scheme with generic point ξ. Let $U \subseteq V \subseteq X$ be non-empty open sets, and let $P \in U$ be any point. Consider the sequence of natural homomorphisms:

$$\mathscr{O}_X(V) \xrightarrow{f} \mathscr{O}_X(U) \xrightarrow{g} \mathscr{O}_{X,P} \xrightarrow{h} \mathscr{O}_{X,\xi} = K(X).$$

Then all maps f, g, and h are **injective**. Consequently, for every open set U, $\mathscr{O}_X(U)$ is an integral domain contained in $K(X)$.

Correspondingly, the function field represents the 'broadest' stalk of an integral scheme. The stalk at any other point can be viewed as a subring of this function field, giving rise to a canonical sequence of injections.

Proof. We verify the injectivity of each map step by step.

1. Injectivity of $h : \mathscr{O}_{X,P} \to \mathscr{O}_{X,\xi}$. Choose an affine open neighborhood $W = \operatorname{Spec} A$ containing P. Then P corresponds to a prime ideal $\mathfrak{p} \subset A$, and the generic point ξ corresponds to the zero ideal (0). The stalks are given by localizations: $\mathscr{O}_{X,P} \cong A_{\mathfrak{p}}$ and $\mathscr{O}_{X,\xi} \cong A_{(0)}$. The map h is the canonical localization map:

$$h : \frac{a}{b} \in A_{\mathfrak{p}} \longmapsto \frac{a}{b} \in A_{(0)}.$$

(where $b \notin \mathfrak{p}$, which implies $b \neq 0$, so $b \in A \setminus (0)$). Suppose $h(a/b) = 0$ in $A_{(0)}$. This means $a/b = 0/1$, which implies $a \cdot 1 = 0$ in the domain A. Thus $a = 0$. Consequently, $a/b = 0$ in $A_{\mathfrak{p}}$. Therefore, h is injective.

2. Injectivity of $g : \mathscr{O}_X(U) \to \mathscr{O}_{X,P}$. Let $S \in \mathscr{O}_X(U)$. The map is defined by taking the germ: $S \mapsto S_P$. Suppose $S_P = 0$. We want to prove $S = 0$, which is equivalent to showing $S_Q = 0$ for all $Q \in U$. Since the map $h_Q : \mathscr{O}_{X,Q} \to \mathscr{O}_{X,\xi}$ is injective (as proved in step 1) for any point Q, it suffices to show that the image of S_Q in $\mathscr{O}_{X,\xi}$ is zero.

Since $S_P = 0$, there exists an open neighborhood $W \subseteq U$ of P such that $S|_W = 0$. Because X is irreducible, the generic point ξ belongs to every non-empty open set, so $\xi \in W$. This implies that the germ at the generic point is zero:

$$S_\xi = (S|_W)_\xi = 0.$$

Now, for **any** point $Q \in U$, the diagram of stalks maps S_Q to S_ξ. Since $S_\xi = 0$ and the map $\mathscr{O}_{X,Q} \to \mathscr{O}_{X,\xi}$ is injective, we must have $S_Q = 0$. Since $S_Q = 0$ for all $Q \in U$, we conclude $S = 0$. Thus g is injective.

3. Injectivity of $f : \mathscr{O}_X(V) \to \mathscr{O}_X(U)$. Let $S \in \mathscr{O}_X(V)$. The map is the restriction: $S \mapsto S|_U$. Suppose $S|_U = 0$. We want to show $S_P = 0$ for all $P \in V$. Consider any point $P \in V$. We have the composition of maps $\mathscr{O}_X(V) \to \mathscr{O}_{X,\xi}$ sending $S \mapsto S_\xi$. From the hypothesis $S|_U = 0$, and since U is non-empty, we have $\xi \in U$. Thus the germ at the generic point is determined by the restriction to U:

$$S_\xi = (S|_U)_\xi = 0.$$

Now, consider the stalk map at P: $\mathcal{O}_{X,P} \to \mathcal{O}_{X,\xi}$. This map sends $S_P \mapsto S_\xi$. We established in Step 1 that this map is injective. Since the image $S_\xi = 0$, the pre-image must be zero:

$$S_P = 0.$$

Since this holds for all $P \in V$, the section S is the zero section. Thus f is injective. □

We now introduce the notion of Cartier divisors for arbitrary schemes. Unlike Weil divisors, which rely on codimension-one points, Cartier divisors are defined via local equations. This definition is robust and works for non-normal and non-Noetherian schemes.

To define "rational functions" on a general scheme X (which may be reducible or have nilpotents), we need a careful definition of non-zero divisors.

Definition 41 Sheaf of Total Quotient Rings.

Let X be a scheme. For any open subset $U \subseteq X$, let $S(U)$ denote the set of elements $s \in \Gamma(U, \mathcal{O}_X)$ such that for every point $x \in U$, the germ s_x is not a zero divisor in the local ring $\mathcal{O}_{X,x}$.

The presheaf $U \mapsto S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ (localization of the ring of sections) is not necessarily a sheaf. We define the **sheaf of total quotient rings** of X, denoted by $\mathcal{K}_X$, to be the sheafification of this presheaf.

Note 42 Function Field for Integral Schemes.

If X is an integral scheme, the situation simplifies significantly. In this case, for any non-empty open affine $U = \operatorname{Spec} A$, $S(U) = A \setminus \{0\}$. The localization is the fraction field $K = \operatorname{Frac}(A)$. Thus, $\mathcal{K}_X$ is the constant sheaf associated to the function field $K(X)$.

Let $\mathcal{K}_X^*$ denote the sheaf of multiplicative groups of invertible elements in $\mathcal{K}_X$. Similarly, let $\mathcal{O}_X^*$ denote the sheaf of invertible elements in $\mathcal{O}_X$. There is a natural inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{K}_X^*$.

Definition 43 Cartier Divisor.

A **Cartier divisor** on a scheme X is a global section of the quotient sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$. We denote the group of Cartier divisors by $\operatorname{Div}(X) := \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$.

Note 44 Local Description of Cartier Divisors.

Unpacking the definition of a section of a quotient sheaf, a Cartier divisor D can be described by an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X and a collection of "local equations" $\{f_i \in \Gamma(U_i, \mathcal{K}_X^*)\}_{i \in I}$ such that for all i, j:

$$\frac{f_i}{f_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*).$$

We write $D = \{(U_i, f_i)\}$. Two collections define the same divisor if their ratio is a unit on the common refinement.

Additively, we define the sum of two divisors $D = \{(U_i, f_i)\}$ and $E = \{(U_i, g_i)\}$ as $D + E = \{(U_i, f_i g_i)\}$. The zero element is the divisor defined by $\{(X, 1)\}$.

On a scheme X, we consider the short exact sequence of sheaves:

$$0 \longrightarrow \mathscr{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow \mathcal{K}_X^*/\mathscr{O}_X^* \longrightarrow 0$$

By applying the global section functor $\Gamma(X, -)$, we obtain the following **left-exact** sequence:

$$0 \longrightarrow \Gamma(X, \mathscr{O}_X^*) \longrightarrow \Gamma(X, \mathcal{K}_X^*) \xrightarrow{\text{div}} \Gamma(X, \mathcal{K}_X^*/\mathscr{O}_X^*) \xrightarrow{\delta} H^1(X, \mathscr{O}_X^*)$$

Definition 45 Principal Cartier Divisor.

A Cartier divisor is called **principal** if it lies in the image of the natural map $\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathscr{O}_X^*)$. If $f \in \Gamma(X, \mathcal{K}_X^*)$ is a global rational function, its associated principal divisor is denoted by $\text{div}(f)$.

Definition 46 Linear Equivalence.

Two Cartier divisors D and E are **linearly equivalent**, denoted $D \sim E$, if their difference $D - E$ is a principal divisor. The group of Cartier divisors modulo linear equivalence is called the **Cartier Class Group** (or Picard Group of divisors):

$$\text{CaCl}(X) := \text{Div}(X)/\sim .$$

The definition of Cartier divisors fits perfectly into the exact sequence of sheaves:

$$0 \longrightarrow \mathscr{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow \mathcal{K}_X^*/\mathscr{O}_X^* \longrightarrow 0.$$

Taking the long exact sequence of cohomology, we obtain:

$$0 \to \Gamma(X, \mathscr{O}_X^*) \to \Gamma(X, \mathcal{K}_X^*) \xrightarrow{\text{div}} \Gamma(X, \mathcal{K}_X^*/\mathscr{O}_X^*) \xrightarrow{\delta} H^1(X, \mathscr{O}_X^*) \to H^1(X, \mathcal{K}_X^*) \dots \tag{5.6}$$

Proposition 47 Cartier Class Group Injects into Picard Group.

For any scheme X, there is an injection of groups:

$$\text{CaCl}(X) \hookrightarrow \text{Pic}(X).$$

Furthermore, if X is integral, we have $\text{CaCl}(X) \cong \text{Pic}(X)$. In particular, $\text{CaCl}(X)$ is isomorphic to the cohomology group $H^1(X, \mathscr{O}_X^*)$.

Proof. We interpret the exact sequence (5.6). The term $\Gamma(X, \mathcal{K}_X^*/\mathscr{O}_X^*)$ is precisely the group of Cartier divisors $\text{Div}(X)$. The map $\text{div} : \Gamma(X, \mathcal{K}_X^*) \to \text{Div}(X)$ sends a global rational function to its principal divisor. Thus, the cokernel of this map is exactly the Cartier Class Group:

$$\text{Coker}(\text{div}) = \text{Div}(X)/\text{Im}(\Gamma(X, \mathcal{K}_X^*)) = \text{CaCl}(X).$$

By the exactness of the sequence, we have an injection induced by the boundary map δ:

$$\delta : \text{CaCl}(X) \hookrightarrow H^1(X, \mathscr{O}_X^*).$$

We know that $H^1(X, \mathscr{O}_X^*) \cong \text{Pic}(X)$ (the group of isomorphism classes of invertible sheaves).

□

Note 48 Boundary Map Interpretation.

The boundary map δ in the proof above assigns to a divisor $D = \{(U_i, f_i)\}$ the 1-cocycle $\{g_{ij}\} \in H^1(\mathcal{U}, \mathcal{O}_X^*)$ defined by $g_{ij} = f_i/f_j$. This cocycle represents the transition functions of the line bundle $\mathcal{L}(D)$.

Given a Cartier divisor $D = \{(U_i, f_i)\}$, we define a subsheaf $\mathcal{L}(D)$ of $\mathcal{K}_X$ by generating it locally via f_i^{-1}:

$$\mathcal{L}(D)|_{U_i} = \mathcal{O}_{U_i} \cdot f_i^{-1}.$$

Since $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$, these local definitions glue correctly to form a globally defined $\mathcal{O}_X$-submodule of $\mathcal{K}_X$. Locally it is generated by a single element, so it is an invertible sheaf.

The map $D \mapsto \mathcal{L}(D)$ induces the isomorphism $\mathrm{CaCl}(X) \cong \mathrm{Pic}(X)$.

We have established that for any scheme, there is an injection $\mathrm{CaCl}(X) \hookrightarrow \mathrm{Pic}(X)$. We now prove that for integral schemes, every invertible sheaf arises from a Cartier divisor, making this map an isomorphism.

Theorem 49 Cartier Divisors and Picard Group for Integral Schemes.

Let X be an integral scheme. Then the natural homomorphism defined by $D \mapsto \mathcal{L}(D)$ is an isomorphism:

$$\mathrm{CaCl}(X) \xrightarrow{\sim} \mathrm{Pic}(X).$$

Proof. Since we already know the map is injective (Proposition 47), we only need to prove surjectivity. That is, given an arbitrary invertible sheaf $\mathcal{L}$, we must find a Cartier divisor D such that $\mathcal{L} \cong \mathcal{L}(D)$. By Proposition 47(1), it suffices to show that $\mathcal{L}$ is isomorphic to **some** invertible subsheaf of the sheaf of total quotient rings $\mathcal{K}_X$.

Step 1: Simplify $\mathcal{K}_X$. Since X is an integral scheme, it is reduced and irreducible. Let ξ be its generic point. The local ring $\mathcal{O}_{X,\xi}$ is a field, which is precisely the function field $K(X)$. For any open set $U \neq \emptyset$, the ring of sections $\mathcal{O}_X(U)$ is a domain contained in $K(X)$. The non-zero divisors are simply non-zero elements. Thus, the sheaf of total quotient rings $\mathcal{K}_X$ is the constant sheaf associated to the field $K(X)$. We denote this constant sheaf by $\underline{K}$.

Step 2: Tensor with the Function Field. Consider the sheaf $\mathcal{L}_K := \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X$. Since $\mathcal{L}$ is locally free of rank 1 (invertible), there exists an open cover $\{U_i\}$ such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$. Restricting the tensor product to U_i, we have:

$$(\mathcal{L} \otimes \mathcal{K}_X)|_{U_i} \cong \mathcal{L}|_{U_i} \otimes_{\mathcal{O}_{U_i}} \mathcal{K}_X|_{U_i} \cong \mathcal{O}_{U_i} \otimes_{\mathcal{O}_{U_i}} \underline{K}|_{U_i} \cong \underline{K}|_{U_i}.$$

Thus, $\mathcal{L}_K$ is a locally constant sheaf of K-vector spaces of dimension 1.

Step 3: Global Trivialization. Since X is integral, it is irreducible as a topological space. A locally constant sheaf on an irreducible space is necessarily constant.

Therefore, we have a global isomorphism:

$$\Phi : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X \xrightarrow{\sim} \mathcal{K}_X.$$

Step 4: Constructing the Embedding. There is a canonical homomorphism $j : \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ defined by $s \mapsto s \otimes 1$. Since $\mathcal{L}$ is locally free (hence flat) and $\mathcal{O}_X \to \mathcal{K}_X$ is injective (as X is integral), the map j is injective. Composing with the isomorphism Φ

from Step 3, we get an injection:

$$\Psi : \mathcal{L} \xrightarrow{j} \mathcal{L} \otimes \mathcal{K}_X \xrightarrow{\Phi} \mathcal{K}_X.$$

The image $\Psi(\mathcal{L})$ is a subsheaf of $\mathcal{K}_X$ which is isomorphic to $\mathcal{L}$. Let $\mathscr{I} = \Psi(\mathcal{L}) \subset \mathcal{K}_X$. This is an invertible subsheaf of $\mathcal{K}_X$. By the correspondence established in the previous section, there exists a Cartier divisor D such that $\mathcal{L}(D) = \mathscr{I} \cong \mathcal{L}$. □

Note 50 Importance of Integrality.

The proof relies heavily on the fact that $\mathcal{K}_X$ is a **constant** sheaf. If X is reducible (e.g., two lines crossing), $\mathcal{K}_X$ is not constant, and there may exist invertible sheaves that are not subsheaves of $\mathcal{K}_X$. In that case, the map $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$ is injective but not necessarily surjective.

While Cartier divisors are defined via local equations, a more geometric intuition comes from counting zeros and poles along codimension-one subvarieties. To rigorous define this counting, we need the local rings at these subvarieties to be DVRs.

Definition 51 Regular in Codimension One.

A scheme X is said to be **regular in codimension one** if for every point $x \in X$ of codimension one (i.e., $\dim \mathscr{O}_{X,x} = 1$), the local ring $\mathscr{O}_{X,x}$ is a regular local ring.

Since a regular local ring of dimension one is precisely a DVR, this condition ensures that we can define valuations along prime divisors.

Note 52 The Condition $(*)$.

In this section, we primarily consider schemes satisfying the following set of conditions, denoted as $(*)$:

1. X is **Noetherian**: To ensure the decomposition into irreducible components behaves well.
2. X is **integral**: So there is a unique function field $K(X)$.
3. X is **separated**: To ensure uniqueness of limits.
4. X is **regular in codimension one**.

The most important examples satisfying $(*)$ are **non-singular varieties** over a field and **Noetherian normal schemes**. Recall that a Noetherian normal domain is integrally closed, and localization of an integrally closed domain is integrally closed. Thus, a dimension one local ring on a normal scheme is a DVR.

Under condition $(*)$, we can define divisors as formal sums of geometric hypersurfaces.

Definition 53 Weil Divisor.

Let X satisfy $(*)$.

1. A **prime divisor** Y on X is a closed integral subscheme of codimension one.
2. A **Weil divisor** D is an element of the free abelian group generated by the prime divisors. We write $D = \sum n_i Y_i$, where $n_i \in \mathbb{Z}$ and the sum is finite.
3. The group of all Weil divisors is denoted by $\mathrm{Div}(X)$.

Definition 54 Valuation of a Function.

Let Y be a prime divisor on X and let $\eta \in Y$ be its generic point. Since X is regular in codimension one, the local ring $\mathcal{O}_{X,\eta}$ is a DVR with quotient field $K(X)$. Let v_Y be the corresponding discrete valuation. For any non-zero rational function $f \in K(X)^*$, $v_Y(f)$ is an integer representing the order of vanishing of f along Y.

Lemma 55 Finiteness of Non-Zero Valuations.

Let X satisfy $(*)$ and let $f \in K(X)^*$ be a non-zero function. Then $v_Y(f) = 0$ for all except finitely many prime divisors Y.

Proof. Let $U = \operatorname{Spec} A$ be an open affine subset where f is regular (i.e., $f \in A$). Then $v_Y(f) \geq 0$ for all Y meeting U. The condition $v_Y(f) > 0$ means $Y \cap U$ is contained in the closed set $V(f)$ defined by the ideal (f). Since X is Noetherian, $V(f)$ contains only finitely many irreducible components of codimension one. A similar argument applies to $1/f$ for the poles. □

This lemma allows us to define the **principal Weil divisor** associated to f:

$$\mathrm{div}(f) = \sum_Y v_Y(f) \cdot Y.$$

The group of Weil divisors modulo principal divisors is called the **Divisor Class Group**, denoted $\mathrm{Cl}(X)$.

We now connect the local equation definition (Cartier) with the geometric sum definition (Weil).

Proposition 56 Cartier vs. Weil.

Let X be an integral, separated, Noetherian scheme satisfying $(*)$, and assume further that X is **locally factorial** (i.e., every local ring $\mathcal{O}_{X,x}$ is a UFD). Then there is a natural isomorphism:

$$\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \xrightarrow{\sim} \mathrm{Div}(X).$$

That is, the group of Cartier divisors is isomorphic to the group of Weil divisors. Under this isomorphism, principal Cartier divisors correspond to principal Weil divisors, and $\mathrm{CaCl}(X) \cong \mathrm{Cl}(X)$.

Proof. **Step 1: Cartier to Weil.** Let $D = \{(U_i, f_i)\}$ be a Cartier divisor. We define the associated Weil divisor $D_W = \sum n_Y Y$ as follows: For each prime divisor Y, choose an index i such that $Y \cap U_i \neq \emptyset$. Let η be the generic point of Y. Since $\eta \in U_i$, f_i is a unit in $K(X)$ but not necessarily in $\mathcal{O}_{X,\eta}$. We define the coefficient:

$$n_Y := v_Y(f_i).$$

If j is another index with $Y \cap U_j \neq \emptyset$, then f_i/f_j is a unit in $\mathcal{O}_{X,\eta}$ (since it is a unit in $\mathcal{O}(U_i \cap U_j)$). Thus $v_Y(f_i) = v_Y(f_j)$, so the map is well-defined.

Step 2: Weil to Cartier. This direction requires the **locally factorial** condition. Let $D = \sum n_Y Y$ be a Weil divisor. Let $x \in X$ be any point. The divisor D induces a Weil divisor D_x on the local scheme Spec $\mathcal{O}_{X,x}$. Since $\mathcal{O}_{X,x}$ is a UFD, every height 1 prime ideal is principal (generated by a single element). It is a standard fact that in a Noetherian UFD, every divisor is principal. Thus, on Spec $\mathcal{O}_{X,x}$, D_x is principal, defined by some $f_x \in K(X)$. This f_x works in an open neighborhood U_x of x. Covering X by such open sets $\{U_x\}$ and functions $\{f_x\}$, we verify that $f_x/f_{x'}$ is a unit on overlaps (since they define the same Weil divisor, their valuations difference is zero everywhere, hence a unit). Thus $\{(U_x, f_x)\}$ defines a Cartier divisor. □

Note 57 Non-locally Factorial Case.

If X is normal but not locally factorial (e.g., the quadric cone $xy = z^2$), there are Weil divisors that are not Cartier. In such cases, $\mathrm{CaCl}(X)$ is a proper subgroup of $\mathrm{Cl}(X)$.

We now explore the profound link between the geometry of divisors and the arithmetic of unique factorization. We will establish an isomorphism between the geometric divisor class group and the algebraic ideal class group, rendering the characterization of UFDs an immediate corollary.

To move between global functions and local valuations, we need a fundamental result from commutative algebra regarding integrally closed domains.

Lemma 58 Algebraic Intersection Theorem.

Let A be a Noetherian integrally closed domain with fraction field K. Then:

$$A = \bigcap_{\mathrm{ht}\mathfrak{p}=1} A_{\mathfrak{p}}$$

where the intersection is taken over all prime ideals of height 1.

Proof. It is clear that $A \subseteq \bigcap A_{\mathfrak{p}}$. We prove the reverse inclusion. Let $x \in \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} A_{\mathfrak{p}}$. Write $x = a/b$ with $a, b \in A$ and $b \neq 0$. Consider the ideal of denominators for x:

$$\mathfrak{a} := (bA : aA) = \{r \in A \mid rx \in A\}.$$

We aim to show that $1 \in \mathfrak{a}$, which implies $x \in A$. Suppose for the sake of contradiction that $\mathfrak{a} \subsetneq A$. Since A is Noetherian, the set of associated primes $\mathrm{Ass}(A/\mathfrak{a})$ is non-empty. Moreover, since $\mathfrak{a}$ is the annihilator of the element $\bar{a} \in A/(b)$, any associated prime $\mathfrak{q} \in \mathrm{Ass}(A/\mathfrak{a})$ is also an associated prime of the principal ideal (b) (regarded as a module $A/(b)$).

In a Noetherian integrally closed domain A, every associated prime of a principal ideal has height 1. (This follows from the fact that A satisfies Serre's condition S_2, or more elementarily, because $A_\mathfrak{q}$ is a DVR if $\operatorname{ht}\mathfrak{q} = 1$ and principal ideals have no embedded components).

Therefore, any such $\mathfrak{q}$ has height 1. By our hypothesis, $x \in A_\mathfrak{q}$. This means $x = c/s$ for some $c \in A, s \in A \setminus \mathfrak{q}$. Thus $sx = c \in A$, which implies $s \in \mathfrak{a}$ by the definition of $\mathfrak{a}$. However, $\mathfrak{q}$ is a prime ideal containing $\mathfrak{a}$, so $\mathfrak{a} \subseteq \mathfrak{q}$. This implies $s \in \mathfrak{q}$, contradicting $s \in A \setminus \mathfrak{q}$. Thus, the assumption $\mathfrak{a} \subsetneq A$ is false. Hence $\mathfrak{a} = A$ and $x \in A$. □

Let A be a Noetherian normal domain. In algebra, the **Ideal Class Group**, denoted $\mathrm{Cl}(A)$, is defined as the group of divisor classes of ideals. Specifically, it is the free abelian group generated by height 1 prime ideals modulo the subgroup of principal ideals.

Theorem 59 Isomorphism of Class Groups.

Let $X = \operatorname{Spec} A$ be a Noetherian normal affine scheme. There is a natural isomorphism between the Weil divisor class group of the scheme and the ideal class group of the ring:

$$\Phi : \mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{Cl}(A).$$

Proof. We construct the map explicitly using the correspondence between prime divisors and height 1 prime ideals.

Construction of Φ: Let $Y \subset X$ be a prime divisor. Since $X = \operatorname{Spec} A$, Y corresponds to a unique prime ideal $\mathfrak{p}$ of height 1. We define the map on generators:

$$\Phi(Y) := [\mathfrak{p}] \in \mathrm{Cl}(A).$$

Extending by linearity, a Weil divisor $D = \sum n_i Y_i$ maps to the ideal class of the symbolic power product $\prod \mathfrak{p}_i^{(n_i)}$.

Well-definedness (Principal maps to Principal): Let $f \in K(X)^*$ be a rational function. The principal divisor is $\operatorname{div}(f) = \sum v_{Y_i}(f) Y_i$. In the algebraic group $\mathrm{Cl}(A)$, this maps to the class generated by the valuations. By the property of DVRs and the algebraic structure of A, the ideal associated to this divisor is exactly the principal fractional ideal (f). Thus, principal divisors map to the zero element in $\mathrm{Cl}(A)$.

Surjectivity: By definition, $\mathrm{Cl}(A)$ is generated by height 1 primes. Each height 1 prime $\mathfrak{p}$ corresponds to a prime divisor $V(\mathfrak{p})$ on X.

Injectivity: Suppose a divisor D maps to the trivial class in $\mathrm{Cl}(A)$. This means the corresponding ideal $\mathfrak{a}$ is principal, say $\mathfrak{a} = (f)$ for some $f \in K$. Then the valuation $v_\mathfrak{p}(f)$ matches the coefficient of Y in D for all height 1 primes. Thus $D = \operatorname{div}(f)$ is a principal divisor in $\operatorname{Div}(X)$. □

With the isomorphism $\mathrm{Cl}(\operatorname{Spec} A) \cong \mathrm{Cl}(A)$ established, the condition for A to be a UFD becomes a purely geometric statement about the triviality of the class group.

Corollary 60 UFD Criterion.

Let A be a Noetherian domain and $X = \operatorname{Spec} A$. The following are equivalent:

1. A is a UFD.
2. X is normal and $\mathrm{Cl}(X) = 0$.

Proof. Recall from commutative algebra that a Noetherian domain A is a UFD if and only if A is integrally closed and every height 1 prime ideal is principal. The condition "every height 1 prime is principal" is equivalent to saying the Ideal Class Group $\mathrm{Cl}(A)$ is trivial. By Theorem 59, we have $\mathrm{Cl}(X) \cong \mathrm{Cl}(A)$. Thus, A is a UFD $\iff$ A is normal and $\mathrm{Cl}(A) = 0 \iff X$ is normal and $\mathrm{Cl}(X) = 0$. □

With the algebraic intersection theorem and the isomorphism of class groups, we can now prove the main theorem relating UFDs to the vanishing of the class group.

Proposition 61 UFD Characterization via Class Group.

Let A be a Noetherian domain. Then A is a UFD if and only if:

1. $X = \operatorname{Spec} A$ is normal (i.e., A is integrally closed), and
2. $\mathrm{Cl}(X) = 0$.

Proof.

($\Rightarrow$) Suppose A is a UFD. It is a standard algebraic fact that UFDs are integrally closed, so X is normal. Furthermore, in a UFD, every prime ideal of height 1 is principal. Let Y be a prime divisor on X. It corresponds to a height 1 prime $\mathfrak{p} = (f)$. Then the principal divisor $\operatorname{div}(f)$ is exactly $1 \cdot Y$. Thus every generator of $\mathrm{Div}(X)$ is principal, so $\mathrm{Cl}(X) = 0$.

($\Leftarrow$) Suppose X is normal and $\mathrm{Cl}(X) = 0$. Since X is normal, A is integrally closed. To show A is a UFD, it suffices to show that every height 1 prime ideal is principal. Let $\mathfrak{p}$ be a prime ideal of height 1, corresponding to a prime divisor Y. Since $\mathrm{Cl}(X) = 0$, there exists a rational function $f \in K$ such that $\operatorname{div}(f) = Y$. This means $v_Y(f) = 1$ and $v_{Y'}(f) = 0$ for all other prime divisors Y'. We claim that $f \in A$ and $\mathfrak{p} = (f)$.

1. Since $v_{Y'}(f) \geq 0$ for all prime divisors (including Y), $f \in A_{\mathfrak{p}'}$ for all height 1 primes $\mathfrak{p}'$. By Proposition 58, $f \in A$.
2. Since $v_Y(f) = 1$, $f \in \mathfrak{p}A_{\mathfrak{p}}$ but $f \notin \mathfrak{p}^2 A_{\mathfrak{p}}$. In fact, f generates the maximal ideal of the DVR $A_{\mathfrak{p}}$.
3. For any $g \in \mathfrak{p}$, $v_Y(g) \geq 1$ and $v_{Y'}(g) \geq 0$. Thus $v_{Y'}(g/f) \geq 0$ for all prime divisors. By Proposition 58 again, $g/f \in A$.

Thus $g \in (f)$, so $\mathfrak{p} \subseteq (f)$. Since $f \in \mathfrak{p}$, we have $\mathfrak{p} = (f)$. Since every height 1 prime is principal, A is a UFD. □

Example 62 Examples of Class Groups.

1. $\operatorname{Spec}\mathbb{Z}$: $\mathrm{Cl}(\operatorname{Spec}\mathbb{Z}) \cong \mathrm{Cl}(\mathbb{Z}) = 0$. $\mathbb{Z}$ is a UFD.
2. $\operatorname{Spec} k[x,y,z]/(xy - z^2)$: This is a normal cone. One can show $\mathrm{Cl}(X) \cong \mathbb{Z}/2\mathbb{Z} \neq 0$. Thus it is not a UFD (indeed $z^2 = xy$ is non-unique factorization). The generator of the class group corresponds to the line $V(y,z)$ on the cone.

We now apply the theory of Weil divisors to the fundamental example of projective space $X = \mathbb{P}^n_k$. The geometry of hypersurfaces in projective space is governed entirely by their degree.

Let $S = k[x_0, \dots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n_k$. Recall that S is a UFD.

Definition 63 Degree of a Prime Divisor.

Let $Y \subset \mathbb{P}^n_k$ be a prime divisor (an irreducible closed subscheme of codimension 1). Y is defined by a homogeneous prime ideal of height 1 in S. Since S is a UFD, this ideal is principal, generated by an irreducible homogeneous polynomial $g \in S$. We define the **degree** of Y, denoted $\deg Y$, to be the degree of the polynomial g.

Definition 64 Degree of a Weil Divisor.

We extend the degree function by linearity to the group of all Weil divisors $\mathrm{Div}(X)$. For a divisor $D = \sum n_i Y_i$, we define:

$$\deg D := \sum n_i \deg Y_i.$$

Let H denote the hyperplane defined by $x_0 = 0$. Since x_0 is a polynomial of degree 1, $\deg H = 1$.

The following proposition completely classifies divisors on projective space up to linear equivalence.

Proposition 65 Divisors on Projective Space.

Let $X = \mathbb{P}^n_k$.

1. If $f \in K(X)^*$ is any non-zero rational function, then $\deg(\mathrm{div}(f)) = 0$.
2. If D is any divisor of degree d, then $D \sim dH$ (linearly equivalent).
3. The degree map induces an isomorphism:

$$\deg : \mathrm{Cl}(X) \xrightarrow{\sim} \mathbb{Z}.$$

Proof. (1) **Degree of Principal Divisors:** A rational function f on $\mathbb{P}^n$ is a quotient $f = g/h$ where $g, h \in S$ are homogeneous polynomials of the same degree, say m. We can factor g into irreducible factors $g = \prod g_i^{n_i}$. Then the divisor of zeros is $(g)_0 = \sum n_i V(g_i)$. The degree is $\deg((g)_0) = \sum n_i \deg(g_i) = \deg(g) = m$. Similarly, $\deg((h)_0) = \deg(h) = m$. Since $\mathrm{div}(f) = (g)_0 - (h)_0$ (zeros minus poles), we have:

$$\deg(\mathrm{div}(f)) = \deg(g) - \deg(h) = m - m = 0.$$

(2) **Linear Equivalence:** Let D be an arbitrary divisor. We can separate positive and

negative coefficients to write $D = D_1 - D_2$, where D_1, D_2 are effective. As shown in the proof of (1), any effective divisor corresponds to a homogeneous polynomial (product of the polynomials defining its prime components). Let g_1 define D_1 and g_2 define D_2. Let $d_1 = \deg g_1$ and $d_2 = \deg g_2$. Then $\deg D = d_1 - d_2$. Let $d = \deg D$. We compare D with dH. The divisor dH corresponds to the polynomial x_0^d. Consider the rational function:

$$f = \frac{g_1}{g_2 x_0^d}.$$

The degree of the numerator is d_1. The degree of the denominator is $d_2 + d = d_2 + (d_1 - d_2) = d_1$. Since the degrees match, f is a well-defined rational function on $\mathbb{P}^n$. The divisor of f is:

$$\operatorname{div}(f) = \operatorname{div}(g_1) - \operatorname{div}(g_2) - \operatorname{div}(x_0^d) = D_1 - D_2 - dH = D - dH.$$

Thus $D - dH$ is a principal divisor, so $D \sim dH$.

(3) **Isomorphism:** The map $\deg : \operatorname{Div}(X) \to \mathbb{Z}$ is surjective (since $\deg H = 1$). By part (1), it maps principal divisors to 0, so it descends to a homomorphism $\operatorname{Cl}(X) \to \mathbb{Z}$. By part (2), if $\deg D = 0$, then $D \sim 0H = 0$, so the kernel is trivial. Thus, $\operatorname{Cl}(X) \cong \mathbb{Z}$, generated by the class of a hyperplane $[H]$. □

Corollary 66 Picard Group of Projective Space.

Using the isomorphism between $\operatorname{Cl}(X)$ and $\operatorname{Pic}(X)$ for integral locally factorial schemes, we have:

$$\operatorname{Pic}(\mathbb{P}_k^n) \cong \mathbb{Z}.$$

The generator is the bundle $\mathscr{O}(1)$ associated to the hyperplane divisor H.

Calculating the Divisor Class Group directly from the definition is often difficult. We now introduce two fundamental tools that allow us to compare the Class Group of a scheme X with its open subsets and its polynomial extensions.

The relationship between the Class Group of a scheme and that of an open subset is controlled by the codimension of the complement.

Proposition 67 Excision Exact Sequence.

Let X satisfy the condition $(*)$ (Noetherian, integral, separated, regular in codimension one). Let $Z \subset X$ be a proper closed subset, and let $U = X \setminus Z$ be the open complement.

1. There is a surjective homomorphism $\operatorname{Cl}(X) \to \operatorname{Cl}(U)$, defined by restricting divisors $D \mapsto D|_U$.
2. If $\operatorname{codim}(Z, X) \geq 2$, then this map is an isomorphism:
$$\operatorname{Cl}(X) \xrightarrow{\sim} \operatorname{Cl}(U).$$
3. If Z is an irreducible closed subset of codimension 1 (i.e., a prime divisor), then there is an exact sequence:
$$\mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Cl}(X) \longrightarrow \operatorname{Cl}(U) \longrightarrow 0,$$
where the first map sends the generator 1 to the class of the divisor Z.

Proof.

(1) **Surjectivity:** Let $D_U = \sum n_i Y_i'$ be a Weil divisor on U. Since U is open and X is integral, the closure $\overline{Y_i'}$ in X is a prime divisor on X. Let $D = \sum n_i \overline{Y_i'}$. Then clearly $D|_U = D_U$. Thus the restriction map is surjective.

(2) **Codimension** ≥ 2**:** The group $\mathrm{Div}(X)$ is the free abelian group generated by codimension 1 subvarieties. If $\mathrm{codim}(Z, X) \geq 2$, then Z contains no codimension 1 subsets. Thus, the prime divisors of X are in one-to-one correspondence with the prime divisors of U (via restriction $Y \mapsto Y \cap U$). Hence $\mathrm{Div}(X) \cong \mathrm{Div}(U)$. Furthermore, the function fields are identical $K(X) = K(U)$, so principal divisors correspond exactly. Thus $\mathrm{Cl}(X) \cong \mathrm{Cl}(U)$.

(3) **Codimension** 1**:** If Z is an irreducible divisor, then the kernel of the surjection $\mathrm{Div}(X) \to \mathrm{Div}(U)$ is the subgroup generated explicitly by Z (since any other prime divisor $Y \neq Z$ meets U and restricts to a generator of $\mathrm{Div}(U)$). Specifically, if $D \in \mathrm{Div}(X)$ restricts to a principal divisor $\mathrm{div}(f)|_U$ on U, then $D - \mathrm{div}(f)$ restricts to 0 on U. This means $D - \mathrm{div}(f)$ is supported entirely on Z, so $D - \mathrm{div}(f) = nZ$ for some integer n. Thus $[D] = n[Z]$ in $\mathrm{Cl}(X)$ modulo the image of principal divisors on U that extend to X. This establishes the exact sequence. □

Example 68 Class Group of Affine and Projective Space.

Let $X = \mathbb{A}^n_k$. Since $A = k[x_1, \dots, x_n]$ is a UFD, $\mathrm{Cl}(\mathbb{A}^n) = 0$. Consider projective space $\mathbb{P}^n$. Let H be the hyperplane at infinity (codimension 1). Then $U = \mathbb{P}^n \setminus H \cong \mathbb{A}^n$. By the exact sequence:

$$\mathbb{Z} \xrightarrow{1 \mapsto [H]} \mathrm{Cl}(\mathbb{P}^n) \to \mathrm{Cl}(\mathbb{A}^n) \to 0.$$

Since $\mathrm{Cl}(\mathbb{A}^n) = 0$, the map $\mathbb{Z} \to \mathrm{Cl}(\mathbb{P}^n)$ is surjective. As calculated before, it is also injective (degree map), confirming $\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$.

The second tool establishes the "homotopy invariance" of the class group, stating that crossing with an affine line does not change the class group.

Proposition 69 $\mathbb{A}^1$-Invariance of Class Group.

Let X satisfy the condition $(*)$. Then the product scheme $X \times \mathbb{A}^1$ also satisfies $(*)$, and there is a natural isomorphism:

$$\mathrm{Cl}(X) \cong \mathrm{Cl}(X \times \mathbb{A}^1).$$

Proof.

Regularity: First, we verify $X \times \mathbb{A}^1$ is regular in codimension one. Codimension one points in $X \times \mathbb{A}^1$ are of two types:

1. **Type 1:** $Y \times \mathbb{A}^1$, where Y is a prime divisor in X. The local ring corresponds to $\mathcal{O}_{X,\eta_Y}[t]$, localized at the prime ideal generated by the uniformizer of Y. This is a DVR.
2. **Type 2:** The closure of a point corresponding to an irreducible polynomial $P(t) \in K(X)[t]$. The local ring is a localization of $K(X)[t]$. Since $K(X)[t]$ is a PID, this is also

a DVR.

Isomorphism: Let $\pi : X \times \mathbb{A}^1 \to X$ be the projection. We define $\pi^* : \mathrm{Cl}(X) \to \mathrm{Cl}(X \times \mathbb{A}^1)$ by $D \mapsto \pi^{-1}(D) = D \times \mathbb{A}^1$.

Injectivity: Suppose $D \in \mathrm{Div}(X)$ and $\pi^* D = \mathrm{div}(f)$ for some $f \in K(X \times \mathbb{A}^1) = K(X)(t)$. Since $\pi^* D$ has support only on "vertical" divisors (Type 1), f cannot have zeros or poles on any "horizontal" divisors (Type 2). Since $K(X)[t]$ is a UFD, any rational function involving t non-trivially would have zeros or poles of Type 2. Thus, $f \in K(X) \subset K(X)(t)$. Consequently, $D = \mathrm{div}(f)$ in X, so $[D] = 0$.

Surjectivity: It suffices to show that any prime divisor Z of Type 2 is linearly equivalent to a divisor of Type 1. Consider Z as a prime ideal in $K(X)[t]$. Since $K(X)[t]$ is a PID, Z is generated by a single polynomial $P(t)$. View $P(t)$ as a rational function on $X \times \mathbb{A}^1$. Its divisor is:

$$\mathrm{div}(P(t)) = 1 \cdot Z + \sum n_i (Y_i \times \mathbb{A}^1).$$

Here, the sum $\sum n_i (Y_i \times \mathbb{A}^1)$ represents the zeros/poles of the coefficients of $P(t)$ considered as functions on X. Thus, $Z = \mathrm{div}(P(t)) - \sum n_i \pi^* Y_i$. In the class group, $[Z] = -\sum n_i [\pi^* Y_i]$, which is in the image of π^*. ☐

Corollary 70 UFD Stability under Polynomial Extension.

If $X = \mathrm{Spec}\, A$ and A is a UFD, then $A[t]$ is a UFD.

Proof. A is a UFD $\iff$ $\mathrm{Cl}(X) = 0$. By the proposition, $\mathrm{Cl}(X \times \mathbb{A}^1) \cong \mathrm{Cl}(X) = 0$. Since $X \times \mathbb{A}^1 = \mathrm{Spec}\, A[t]$, this implies $A[t]$ is a UFD. ☐

Using the tools developed above—specifically the excision sequence and $\mathbb{A}^1$-invariance—we can now compute the divisor class group of a product with projective space. This result is fundamental for computing class groups of blow-ups and projective bundles.

Proposition 71 Class Group of Product with Projective Space.

Let X be a scheme satisfying condition $(*)$ (Noetherian, integral, separated, regular in codimension one). Then $X \times \mathbb{P}^n$ also satisfies $(*)$, and there is a natural isomorphism:

$$\mathrm{Cl}(X \times \mathbb{P}^n) \cong \mathrm{Cl}(X) \oplus \mathbb{Z}.$$

The subgroup $\mathrm{Cl}(X)$ is embedded via the pullback of the projection $\pi : X \times \mathbb{P}^n \to X$, and the factor $\mathbb{Z}$ is generated by the class of the divisor $Z = X \times H$, where $H \subset \mathbb{P}^n$ is a hyperplane.

Proof. Let H be the hyperplane in $\mathbb{P}^n$ defined by $x_0 = 0$. Let $Z = X \times H$ be the closed subscheme of $X \times \mathbb{P}^n$. Since H is a prime divisor in $\mathbb{P}^n$, Z is a prime divisor in $X \times \mathbb{P}^n$ (it is integral and of codimension 1). The complement of Z is:

$$U = (X \times \mathbb{P}^n) \setminus (X \times H) \cong X \times (\mathbb{P}^n \setminus H) \cong X \times \mathbb{A}^n.$$

We apply the Excision Sequence (Proposition 67) to the triple $(X \times \mathbb{P}^n, U, Z)$:

$$\mathbb{Z} \xrightarrow{\alpha} \mathrm{Cl}(X \times \mathbb{P}^n) \xrightarrow{\beta} \mathrm{Cl}(X \times \mathbb{A}^n) \longrightarrow 0.$$

Here, the map α is defined by $1 \mapsto [Z]$.

Step 1: Simplify the base. By applying the homotopy invariance property (Proposition 69) n times (since $\mathbb{A}^n = \mathbb{A}^1 \times \cdots \times \mathbb{A}^1$), we have an isomorphism:

$$\mathrm{Cl}(X \times \mathbb{A}^n) \cong \mathrm{Cl}(X).$$

Let $\pi : X \times \mathbb{P}^n \longrightarrow X$ be the natural projection. The isomorphism above is induced by the restriction of π to U, but essentially it identifies the class group of the open set with the base. Substituting this into our sequence, we get:

$$\mathbb{Z} \xrightarrow{\alpha} \mathrm{Cl}(X \times \mathbb{P}^n) \xrightarrow{\beta'} \mathrm{Cl}(X) \longrightarrow 0.$$

Step 2: Splitting the sequence. To show this sequence splits, we define a map in the reverse direction (a section). Consider the projection $\pi : X \times \mathbb{P}^n \longrightarrow X$. This is a flat morphism. The pullback of divisors $\pi^* : \mathrm{Cl}(X) \to \mathrm{Cl}(X \times \mathbb{P}^n)$ is a well-defined homomorphism. (Explicitly, if D is a prime divisor on X, $\pi^* D = D \times \mathbb{P}^n$ is a prime divisor on the product). We claim that $\beta' \circ \pi^* = \mathrm{id}_{\mathrm{Cl}(X)}$. Let $D \in \mathrm{Cl}(X)$. The element $\pi^* D$ is a divisor on $X \times \mathbb{P}^n$ "vertical" to the base. When we restrict this to $U \cong X \times \mathbb{A}^n$, we get $D \times \mathbb{A}^n$. Under the homotopy isomorphism $\mathrm{Cl}(X \times \mathbb{A}^n) \cong \mathrm{Cl}(X)$, this maps back to D. Since $\beta' \circ \pi^* = \mathrm{id}$, the sequence splits. Thus:

$$\mathrm{Cl}(X \times \mathbb{P}^n) \cong \mathrm{Im}(\pi^*) \oplus \mathrm{Im}(\alpha) \cong \mathrm{Cl}(X) \oplus \mathrm{Im}(\alpha).$$

Step 3: Injectivity of α. We must show that $\alpha(1) = [Z]$ has infinite order and does not intersect the image of π^*. Consider the second projection $p_2 : X \times \mathbb{P}^n \longrightarrow \mathbb{P}^n$. This induces a map $p_2^* : \mathrm{Cl}(\mathbb{P}^n) \longrightarrow \mathrm{Cl}(X \times \mathbb{P}^n)$. We know $\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$, generated by $[H]$. Notice that $Z = p_2^{-1}(H) = p_2^*[H]$. If $n \cdot [Z] = 0$ in $\mathrm{Cl}(X \times \mathbb{P}^n)$, then restricting to a fiber over a point $x \in X$ (which is isomorphic to $\mathbb{P}^n$) would imply $n \cdot [H] = 0$ in $\mathrm{Cl}(\mathbb{P}^n_k)$, which implies $n = 0$. Thus $\mathrm{Im}(\alpha) \cong \mathbb{Z}$.

Combining the splitting and the kernel analysis, we obtain:

$$\mathrm{Cl}(X \times \mathbb{P}^n) \cong \mathrm{Cl}(X) \oplus \mathbb{Z}.$$

□

Corollary 72 Picard Group of Projective Space over a UFD.

Let A be a Noetherian UFD (e.g., $A = \mathbb{Z}$). Then:

$$\mathrm{Pic}(\mathbb{P}^n_A) \cong \mathbb{Z}.$$

Proof. Since A is a UFD, $\mathrm{Cl}(\mathrm{Spec}\, A) = 0$. By Proposition 71, $\mathrm{Cl}(\mathbb{P}^n_A) \cong \mathrm{Cl}(\mathrm{Spec}\, A) \times \mathbb{Z} \cong \mathbb{Z}$. Since $\mathbb{P}^n_A$ is regular (smooth over A), the class group is isomorphic to the Picard group. □

5.6 Examples

Example 73 The Affine Line with Two Origins.

Let k be a field. Consider two copies of the affine line, $X_1 = \operatorname{Spec} k[t]$ and $X_2 = \operatorname{Spec} k[u]$. Let $U_1 = D(t) \subset X_1$ and $U_2 = D(u) \subset X_2$ be the open subsets where the coordinates are non-zero. We construct a scheme X by gluing X_1 and X_2 along the isomorphism $\phi : U_1 \to U_2$ induced by the ring isomorphism $k[u, u^{-1}] \to k[t, t^{-1}]$ sending $u \mapsto t$.

We assert that the diagonal morphism $\Delta : X \to X \times_k X$ is not a closed immersion.

A morphism of schemes is a closed immersion if and only if it is a homeomorphism onto a closed subset of the target and the induced map on sheaves is surjective. A necessary condition is that the image of the morphism must be closed in the target. We examine the image of Δ locally.

Consider the affine open subset of the product $X \times_k X$ given by $W = X_1 \times_k X_2 \cong \operatorname{Spec}(k[t] \otimes_k k[u]) \cong \operatorname{Spec} k[t, u] \cong \mathbb{A}^2_k$. The preimage of W under the diagonal morphism Δ is the intersection of X_1 and X_2 in X, which is exactly the locus where the gluing occurred: $\Delta^{-1}(W) = X_1 \cap X_2 \cong \mathbb{A}^1_k \setminus \{0\}$. The restriction of the diagonal morphism to this open set, denoted $\Delta|_W$, corresponds to the ring homomorphism $\psi : k[t, u] \to k[t, t^{-1}]$ defined by $t \mapsto t$ and $u \mapsto t$.

The kernel of ψ is the ideal generated by $(t - u)$, but since the target is localized at t, the image of the morphism of schemes corresponds to the locally closed subset $V(t - u) \cap D(t)$ inside $\mathbb{A}^2_k$. Set-theoretically, the image $\Delta(X) \cap W$ consists of points $\{(a, a) \in \mathbb{A}^2 \mid a \neq 0\}$. The closure of this set in the affine plane W clearly includes the origin $(0, 0)$, which corresponds to the point pair $(0_1, 0_2) \in X_1 \times X_2$. However, $(0, 0)$ is not contained in the image of Δ, as the origins 0_1 and 0_2 are distinct points in X and map to $(0_1, 0_1)$ and $(0_2, 0_2)$ respectively, neither of which lies in the chart $X_1 \times X_2$.

Since the image $\Delta(X)$ is not closed within the open set W, it is not closed in $X \times_k X$. Therefore, Δ is not a closed immersion.

Example 74 The Punctured Affine Plane.

Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$ be the affine plane over a field k. Consider the open subscheme $U = X \setminus \{(0, 0)\}$. The scheme U is not affine.

We proceed by contradiction. Assume that U is an affine scheme. Then U is isomorphic to the spectrum of its ring of global sections, i.e., $U \cong \operatorname{Spec} \Gamma(U, \mathcal{O}_U)$. We compute this ring explicitly.

The open set U admits a standard affine open cover given by $U_x = D(x) = \operatorname{Spec} k[x, y]_x$ and $U_y = D(y) = \operatorname{Spec} k[x, y]_y$. The global sections are determined by the sheaf axiom on this cover:

$$\Gamma(U, \mathcal{O}_U) = \ker\left(\Gamma(U_x, \mathcal{O}_U) \times \Gamma(U_y, \mathcal{O}_U) \to \Gamma(U_x \cap U_y, \mathcal{O}_U)\right).$$

Algebraically, this is the intersection of subrings $k[x, y, x^{-1}] \cap k[x, y, y^{-1}]$ inside the function field $k(x, y)$. This intersection consists of rational functions where the denominator can only involve powers of x (from the first ring) and simultaneously only powers of y (from the second ring). Since $k[x, y]$ is a unique factorization domain and x, y are coprime irreducibles, any element in the intersection must have a trivial denominator. This is a manifestation of the algebraic Hartogs' Lemma, yielding $\Gamma(U, \mathscr{O}_U) \cong k[x, y]$.

If U were affine, the canonical morphism $\pi : U \longrightarrow \operatorname{Spec}\Gamma(U, \mathscr{O}_U) \cong \operatorname{Spec} k[x, y] = \mathbb{A}^2_k$ would be an isomorphism. However, the inclusion morphism $j : U \hookrightarrow \mathbb{A}^2_k$ induces the identity on global sections, so π coincides with j. The morphism j is clearly not surjective (it misses the origin) and thus cannot be an isomorphism. This contradiction demonstrates that U is not affine.

Example 75 Hierarchy of Local Structures.

Let X be a scheme and $x \in X$ a point. We denote the local ring at x by $(\mathscr{O}_{X,x}, \mathfrak{m}_x)$ and the residue field by $k(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$. Let $\widehat{\mathscr{O}}_{X,x}$ denote the $\mathfrak{m}_x$-adic completion of the local ring. There exists a natural sequence of morphisms of affine schemes:

$$\operatorname{Spec} k(x) \xrightarrow{i} \operatorname{Spec} \widehat{\mathscr{O}}_{X,x} \xrightarrow{c} \operatorname{Spec} \mathscr{O}_{X,x} \xrightarrow{j} X.$$

We characterize these morphisms as follows:

The morphism $j : \operatorname{Spec} \mathscr{O}_{X,x} \longrightarrow X$ is the canonical morphism associated with the localization of a scheme. Topologically, j is a homeomorphism onto the set of generalizations of x, denoted by $S_x = \{y \in X \mid x \in \overline{\{y\}}\}$. This is simply the intersection of all open neighborhoods of x in X. It essentially "zooms in" on the germ of the space near x, discarding all geometry not incident to x.

The morphism $c : \operatorname{Spec} \widehat{\mathscr{O}}_{X,x} \longrightarrow \operatorname{Spec} \mathscr{O}_{X,x}$ is induced by the completion homomorphism $\mathscr{O}_{X,x} \longrightarrow \widehat{\mathscr{O}}_{X,x}$. Since the completion of a Noetherian local ring is a faithfully flat extension, c is a faithfully flat morphism. Geometrically, this represents passing from the Zariski local geometry (algebraic functions) to the formal local geometry (power series). If X is a curve over $\mathbb{C}$ and x is a smooth point, this is analogous to passing from algebraic functions defined near x to holomorphic germs (analytic disk).

The morphism $i : \operatorname{Spec} k(x) \longrightarrow \operatorname{Spec} \widehat{\mathscr{O}}_{X,x}$ corresponds to the projection $\widehat{\mathscr{O}}_{X,x} \longrightarrow \widehat{\mathscr{O}}_{X,x}/\widehat{\mathfrak{m}}_x \cong k(x)$. This is a closed immersion identifying the unique closed point of the local scheme.

Collectively, these morphisms trace the "descent" from the global variety X, to the local algebraic neighborhood, to the formal neighborhood (infinitesimal structure), and finally to the geometric point itself.

Example 76 Divisor Class Group of Affine Space.

Let k be a field and let $X = \mathbb{A}^n_k = \operatorname{Spec}(A)$ where $A = k[x_1, \ldots, x_n]$. We assert that the divisor class group $\operatorname{Cl}(X)$ is trivial.

The ring $A = k[x_1, \ldots, x_n]$ is a unique factorization domain (UFD). Since X is the spectrum of a Noetherian domain, the group of Weil divisors $\operatorname{Div}(X)$ is the free abelian group generated by prime divisors, which correspond to prime ideals of height 1 in A.

Let $D \in \operatorname{Div}(X)$ be a Weil divisor. We may write $D = \sum_{i=1}^{m} n_i[Y_i]$, where each Y_i is an integral closed subscheme of codimension 1. Correspondingly, let $\mathfrak{p}_i$ be the prime ideal associated with the generic point of Y_i. Since A is a UFD, every prime ideal of height 1 is principal. Thus, for each i, there exists an irreducible element $f_i \in A$ such that $\mathfrak{p}_i = (f_i)$.

Consider the rational function $f = \prod_{i=1}^m f_i^{n_i} \in K(X)^*$, where $K(X)$ denotes the function field of X. The principal divisor associated to f is exactly $\operatorname{div}(f) = \sum_{i=1}^m n_i \operatorname{ord}_{Y_i}(f)[Y_i]$. Since $\mathfrak{p}_i$ is generated by f_i, the valuation $\operatorname{ord}_{Y_i}(f_i)$ equals 1, and for $j \neq i$, $\operatorname{ord}_{Y_j}(f_i) = 0$. Consequently, $\operatorname{div}(f) = \sum n_i[Y_i] = D$.

This demonstrates that every Weil divisor on $\mathbb{A}_k^n$ is principal. The divisor class group, defined as the quotient of the group of Weil divisors by the subgroup of principal divisors, is therefore trivial. That is, $\operatorname{Cl}(\mathbb{A}_k^n) = 0$.

Example 77 Divisor Class Group of the Ring of Integers in a Number Field.

Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. Consider the affine scheme $X = \operatorname{Spec}(\mathcal{O}_K)$. We show that the divisor class group $\operatorname{Cl}(X)$ is isomorphic to the ideal class group of $\mathcal{O}_K$, denoted by C_K.

The ring $\mathcal{O}_K$ is a Dedekind domain. For a Dedekind domain, the theory of Weil divisors is equivalent to the theory of fractional ideals. Specifically, a prime divisor on X corresponds to a non-zero prime ideal $\mathfrak{p} \subset \mathcal{O}_K$. The group of Weil divisors $\operatorname{Div}(X)$ is the free abelian group on the set of non-zero prime ideals. By the unique factorization of ideals in a Dedekind domain, there is a natural isomorphism between $\operatorname{Div}(X)$ and the group of fractional ideals I_K. This isomorphism maps a divisor $D = \sum n_{\mathfrak{p}}[\mathfrak{p}]$ to the fractional ideal $\prod \mathfrak{p}^{n_{\mathfrak{p}}}$.

Under this isomorphism, a principal divisor $\operatorname{div}(f)$ for $f \in K^*$ corresponds to the principal fractional ideal $(f) = f\mathcal{O}_K$. The divisor class group $\operatorname{Cl}(X)$ is defined as $\operatorname{Div}(X)/\sim$, where the equivalence relation is modulo principal divisors. Similarly, the ideal class group C_K is the quotient of the group of fractional ideals by the subgroup of principal fractional ideals.

Since the map $\operatorname{Div}(X) \to I_K$ is an isomorphism that maps the subgroup of principal divisors onto the subgroup of principal fractional ideals, it induces an isomorphism on the quotients. Therefore, $\operatorname{Cl}(\operatorname{Spec}(\mathcal{O}_K)) \cong C_K$.

Example 78 Class Group vs. Cartier Class Group on a Quadric Cone.

Let k be an algebraically closed field of characteristic not equal to 2. Consider the quadric cone $X \subset \mathbb{A}_k^3$ defined by the equation $z^2 = xy$. Let $A = k[x, y, z]/(xy - z^2)$, so $X = \operatorname{Spec}(A)$. We demonstrate that $\operatorname{Cl}(X) \cong \mathbb{Z}/2\mathbb{Z}$ and that $\operatorname{CaCl}(X) = 0$.

We first compute the Weil divisor class group $\operatorname{Cl}(X)$. We utilize the excision exact sequence for class groups. Let U be the open subset $D(x) \subset X$. The coordinate ring of U is the localization $A_x = k[x, y, z, x^{-1}]/(xy - z^2)$. Within this ring, we have the relation $y = z^2x^{-1}$, which allows us to eliminate y. Thus, $A_x \cong k[x, x^{-1}, z]$. Since $k[x, x^{-1}, z]$ is a localization of a polynomial ring, it is a UFD, implying $\operatorname{Cl}(U) = 0$.

The complement $Z = X \setminus U$ is the closed set defined by $V(x)$. In the ring A, the relation $xy = z^2$ implies that if $x = 0$, then $z^2 = 0$, and thus $z = 0$ (since the nilradical is trivial). The coordinate y remains unconstrained. Therefore, Z is the line defined by the ideal $\mathfrak{p} = (x, z)$, which is a prime ideal of height 1. This corresponds to a prime divisor $L = V(x, z)$.

According to the standard exact sequence for class groups:

$$\mathbb{Z} \xrightarrow{\phi} \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0,$$

where the map ϕ sends 1 to the class of the closure of the divisor restricted to U. However, since Z is an irreducible component of the complement, the sequence is effectively generated by the class $[L]$ in $\mathrm{Cl}(X)$. Specifically, $\mathrm{Cl}(X)$ is generated by $[L]$.

To determine the order of $[L]$, we examine the principal divisor associated with x. In the ring A, x vanishes along L. We compute the order of vanishing of x along L. In the localization $A_{\mathfrak{p}}$, $\mathfrak{p}A_{\mathfrak{p}}$ is the maximal ideal. Since $x = z^2/y$ and y is a unit in the local ring at a generic point of L (where $y \neq 0$), we see that x scales as z^2. This suggests that $\mathrm{div}(x) = 2[L]$. Thus $2[L] = 0$ in $\mathrm{Cl}(X)$.

Since A is not a UFD (the relation $xy = z^2$ exhibits two distinct factorizations into irreducibles), the class $[L]$ is non-trivial. Therefore, $\mathrm{Cl}(X) \cong \mathbb{Z}/2\mathbb{Z}$.

Next, we consider the Cartier divisor class group $\mathrm{CaCl}(X)$, which is naturally isomorphic to $\mathrm{Pic}(X)$. There is a canonical injection $\mathrm{CaCl}(X) \hookrightarrow \mathrm{Cl}(X)$ for normal schemes, mapping a locally principal divisor to its Weil class. The image of this map consists of those Weil divisors which are locally principal. The generator of $\mathrm{Cl}(X)$ is $[L]$, corresponding to the ideal $\mathfrak{p} = (x, z)$.

We examine the local ring at the vertex of the cone, $\mathfrak{m} = (x, y, z)$. In the local ring $A_{\mathfrak{m}}$, the ideal $\mathfrak{p}A_{\mathfrak{m}}$ is not principal. This is because the embedding dimension of X at the origin is 3, while the dimension of X is 2; the origin is a singular point. If $\mathfrak{p}$ were locally principal at the origin, the geometry would require the singularity to be resolved or L to avoid it. Since L passes through the singular vertex and is not cut out by a single equation locally (it requires $x = 0$ and $z = 0$), $[L]$ is not a Cartier divisor.

Consequently, the subgroup of Cartier divisors inside $\mathrm{Cl}(X)$ contains only the identity element. Thus, $\mathrm{CaCl}(X) = 0$. This discrepancy reflects the fact that the cone is normal but not locally factorial at the cone point.

CHAPTER 6

Discussions on Selected Topics in Scheme Theory

6.1 Separated Morphisms and Quasi-compact Morphisms

Definition 1 Topological Properties of Morphisms.

Let $f : X \to Y$ be a morphism of schemes. The diagonal morphism is denoted by $\Delta : X \to X \times_Y X$.

(i) **Quasi-compact Morphism:** The morphism f is called *quasi-compact* if the inverse image of every quasi-compact open subset of Y is quasi-compact in X.

(ii) **Separated Morphism:** The morphism f is called *separated* if the diagonal morphism Δ is a **closed immersion**. A scheme X is separated over a base S if the structure map $X \to S$ is separated.

(iii) **Quasi-separated Morphism:** The morphism f is called *quasi-separated* if the diagonal morphism Δ is **quasi-compact**.

(iv) **QCQS:** A morphism is called *QCQS* if it is both quasi-compact and quasi-separated.

Proposition 2 Separatedness of Affine Morphisms.

Let $f : X \to Y$ be a morphism of affine schemes. Then f is separated.

Proof. Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. The morphism f corresponds to a ring homomorphism $\phi : A \to B$. The product scheme $X \times_Y X$ is affine, specifically given by $\operatorname{Spec}(B \otimes_A B)$. The diagonal morphism $\Delta : X \to X \times_Y X$ corresponds algebraically to the multiplication map $\mu : B \otimes_A B \to B$, defined by $b_1 \otimes b_2 \mapsto b_1 b_2$. Since μ is clearly a surjective ring homomorphism (mapping $b \otimes 1$ to b), it induces a closed immersion of schemes. Since the diagonal is a closed immersion, the morphism f is separated. □

Proposition 3 The Diagonal is an Immersion.

For any morphism of schemes $f : X \to Y$, the diagonal morphism $\Delta : X \to X \times_Y X$ is an immersion.

Proof. To show that Δ is an immersion, we must show that it is a closed immersion into an open subscheme of $X \times_Y X$. This is a local question on the target. Let $\{V_i\}$ be an affine open covering of Y. For each i, let $\{U_{ij}\}$ be an affine open covering of $f^{-1}(V_i)$. The collection $\{U_{ij} \times_{V_i} U_{ij}\}$ forms an open covering of the image of the diagonal in $X \times_Y X$, and these sets are open in $X \times_Y X$. The restriction of the diagonal morphism to each U_{ij} is the map $\Delta_{ij} : U_{ij} \to U_{ij} \times_{V_i} U_{ij}$. Since U_{ij} and V_i are affine, this is a morphism between affine schemes. By the previous proposition, Δ_{ij} is a closed immersion. Since Δ is locally a closed immersion, it is by definition an immersion. (Note: This implies the image $\Delta(X)$ is locally closed in $X \times_Y X$). □

Proposition 4 Intersection of Affines (Separated Case).

Let S be an affine scheme, and let X be a separated scheme over S. If U and V are affine open subsets of X, then their intersection $U \cap V$ is also affine.

Proof. Consider the product $X \times_S X$. Since S, U, and V are affine, the product $U \times_S V$ is an affine open subset of $X \times_S X$. We can identify the intersection $U \cap V$ with the fiber product of the diagonal:

$$U \cap V \cong \Delta^{-1}(U \times_S V) = X \times_{(X \times_S X)} (U \times_S V).$$

Since X is separated over S, the diagonal morphism $\Delta : X \to X \times_S X$ is a closed immersion. The property of being a closed immersion is stable under base change. Therefore, the induced map $U \cap V \to U \times_S V$ is a closed immersion. This exhibits $U \cap V$ as a closed subscheme of the affine scheme $U \times_S V$. A closed subscheme of an affine scheme is uniquely determined by a quotient of its coordinate ring, hence it is affine. Thus $U \cap V$ is affine. □

Proposition 5 Intersection of Affines (Quasi-separated Case).

Let X be a quasi-separated scheme. If U and V are two quasi-compact open subsets of X (e.g., affine opens), then their intersection $U \cap V$ is quasi-compact. In particular, the intersection of two affine open sets can be covered by finitely many affine open sets.

Proof. Similar to the separated case, we utilize the diagonal morphism $\Delta : X \to X \times X$ (over $\operatorname{Spec} \mathbb{Z}$ or the base). We have the identification $U \cap V \cong \Delta^{-1}(U \times V)$. If U and V are quasi-compact (which is true if they are affine), then $U \times V$ is a quasi-compact open subset of $X \times X$. By the definition of a quasi-separated scheme, the diagonal morphism Δ is quasi-compact. This means the inverse image of any quasi-compact open set is quasi-compact. Therefore, $\Delta^{-1}(U \times V)$ is quasi-compact. Since $U \cap V$ is quasi-compact, and the affine open sets form a basis for the topology, $U \cap V$ can be covered by a finite number of affine open subsets. □

Proposition 6 Preservation of Quasi-coherence and Coherence.

Let $f : X \to Y$ be a morphism of schemes.

(i) **Pullback of Quasi-coherent Sheaves:** If $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_Y$-module, then the inverse image $f^*\mathscr{G}$ is a quasi-coherent $\mathscr{O}_X$-module.

(ii) **Pullback of Coherent Sheaves:** If X and Y are **Noetherian** schemes and $\mathscr{G}$ is a coherent $\mathscr{O}_Y$-module, then $f^*\mathscr{G}$ is a coherent $\mathscr{O}_X$-module.

(iii) **Pushforward of Quasi-coherent Sheaves (QCQS):** If f is a **quasi-compact** and **quasi-separated** morphism, and $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module, then the direct image $f_*\mathscr{F}$ is a quasi-coherent $\mathscr{O}_Y$-module.

Note 7 Pushforward of Coherent Sheaves.

It is crucial to note that if $\mathscr{F}$ is coherent, $f_*\mathscr{F}$ is **not necessarily coherent**, even if X and Y are Noetherian.

Counter-example: Consider the structural morphism $f : \mathbb{A}^1_k \to \operatorname{Spec} k$ for a field k. The structure sheaf $\mathscr{O}_{\mathbb{A}^1}$ is coherent. However, the global sections $f_*\mathscr{O}_{\mathbb{A}^1} \cong k[t]$ is an infinite-dimensional vector space over k, hence not a coherent (finitely generated) module over $\operatorname{Spec} k$. For coherence to be preserved, one typically requires f to be a *proper morphism* (Theorem of Finiteness for Proper Morphisms).

Proof. The verification is local on the base Y. We may assume $Y = \operatorname{Spec} A$ is an affine scheme.

Proof of (i): Pullback of QCoh. Since $\mathscr{G}$ is quasi-coherent on the affine scheme $Y = \operatorname{Spec} A$, it corresponds to an A-module M, i.e., $\mathscr{G} \cong \widetilde{M}$. Let $U \subseteq X$ be an affine open subset, say $U = \operatorname{Spec} B$. The restriction of the morphism f gives a ring homomorphism $\phi : A \to B$. By the definition of pullback for modules, the restriction of $f^*\mathscr{G}$ to U is:

$$(f^*\mathscr{G})|_U \cong \widetilde{B \otimes_A M}.$$

Since $B \otimes_A M$ is a B-module, $(f^*\mathscr{G})|_U$ is a quasi-coherent sheaf on U. Since X can be covered by such affine open sets, and quasi-coherence is a local property, $f^*\mathscr{G}$ is quasi-coherent on X.

Proof of (ii): Pullback of Coh. We retain the notation from (i). Since Y is Noetherian, coherence of $\mathscr{G}$ implies that M is a **finitely generated** A-module. Since X is Noetherian, the affine ring B is Noetherian. The tensor product $B \otimes_A M$ is generated over B by the elements $1 \otimes m_i$ (where m_i generate M). Thus, $B \otimes_A M$ is a finitely generated module over the Noetherian ring B. This implies that $(f^*\mathscr{G})|_U$ is a coherent sheaf. Since coherence is local on a Noetherian scheme, $f^*\mathscr{G}$ is coherent.

Proof of (iii): Pushforward of QCoh (The QCQS Case). This is the most subtle part. We assume $Y = \operatorname{Spec} A$.

- **Step 1: Finite Affine Cover.** Since f is **quasi-compact**, the preimage $f^{-1}(Y) = X$ is quasi-compact. Thus, we can cover X by a **finite** number of affine open subsets $\{U_i\}_{i \in I}$, where $U_i = \operatorname{Spec} B_i$.

- **Step 2: Intersection Property.** Since f is **quasi-separated**, the intersection of any two affine opens $U_i \cap U_j$ is quasi-compact. Thus, each $U_i \cap U_j$ can be covered by finitely many affine open sets $\{U_{ijk}\}_{k\in K_{ij}}$, where $U_{ijk} = \operatorname{Spec} C_{ijk}$.

For any open subset $V \subseteq Y$, the sheaf axiom for $\mathscr{F}$ on the cover $\{f^{-1}(V) \cap U_i\}$ of $f^{-1}(V)$ gives an exact sequence:

$$0 \to \mathscr{F}(f^{-1}(V)) \to \prod_{i\in I} \mathscr{F}(f^{-1}(V) \cap U_i) \to \prod_{i,j\in I} \mathscr{F}(f^{-1}(V) \cap U_i \cap U_j).$$

This holds for any open V. Therefore, we have an exact sequence of sheaves on Y:

$$0 \longrightarrow f_*\mathscr{F} \longrightarrow \bigoplus_{i\in I} f_*(\mathscr{F}|_{U_i}) \longrightarrow \bigoplus_{i,j,k} f_*(\mathscr{F}|_{U_{ijk}}).$$

(Note: Since the index sets are finite, the direct product corresponds to the direct sum).

Now consider the terms in this sequence:

- The term $f_*(\mathscr{F}|_{U_i})$ is the pushforward of a quasi-coherent sheaf from an affine scheme $U_i = \operatorname{Spec} B_i$ to an affine scheme $Y = \operatorname{Spec} A$. If $\mathscr{F}|_{U_i} \cong \widetilde{N_i}$ for a B_i-module N_i, then the pushforward is simply the sheaf associated to N_i viewed as an A-module (via restriction of scalars). Thus, it is quasi-coherent on Y.
- Similarly, the third term involves pushforwards from affine U_{ijk} to affine Y, which are also quasi-coherent.

The category of quasi-coherent sheaves is an abelian category closed under kernels. Since $f_*\mathscr{F}$ is the kernel of a morphism between two quasi-coherent sheaves (finite direct sums of quasi-coherent sheaves), $f_*\mathscr{F}$ must be quasi-coherent. □

6.2 Sheaves of Ideals and Support

Intuition 8 Sheaves of Ideals as Sheaves of Modules.

Just as an ideal of a ring can be viewed as a special kind of module, we now begin to discuss the properties of a sheaf of ideals (on a scheme) as a particular class of sheaves of modules.

Understanding how ideal sheaves behave under morphisms is crucial for defining the preimage of closed subschemes.

Definition 9 Ideal Sheaf Operations.

Let $(X, \mathscr{O}_X)$ be a ringed space.

1. Let $Z \subseteq X$ be a closed subscheme. The ideal sheaf of Z, usually denoted by $\mathscr{I}_Z$, is the kernel of the restriction map (morphism of sheaves) from the structure sheaf of

X to the structure sheaf of Z:

$$0 \to \mathscr{I}_Z \to \mathscr{O}_X \to i_*\mathscr{O}_Z \to 0$$

Where $i : Z \hookrightarrow X$ is the closed immersion.

2. Let $\mathscr{I}$ be a sheaf of ideals of $\mathscr{O}_X$ and $\mathscr{F}$ an $\mathscr{O}_X$-module. We define the submodule $\mathscr{I}\mathscr{F} \subseteq \mathscr{F}$ as the image of the natural tensor map:

$$\mathscr{I}\mathscr{F} := \operatorname{im}(\mathscr{I} \otimes_{\mathscr{O}_X} \mathscr{F} \longrightarrow \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{F} \cong \mathscr{F}).$$

3. Let $f : X \to Y$ be a morphism of ringed spaces and $\mathcal{J}$ a sheaf of ideals on Y. The **inverse image ideal sheaf**, denoted by $\mathcal{J}\mathscr{O}_X$, is the sheaf of ideals on X generated by the image of the map $f^{-1}\mathcal{J} \to f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. Explicitly:

$$\mathcal{J}\mathscr{O}_X := \operatorname{im}(f^*\mathcal{J} \longrightarrow f^*\mathscr{O}_Y \cong \mathscr{O}_X).$$

If you look at an affine open set $U = \operatorname{Spec}(A) \subseteq X$, the closed subscheme $Z \cap U$ corresponds to an ideal $I \subseteq A$. On this open set, the ideal sheaf is simply the sheaf associated to the module I:

$$\mathscr{I}_Z|_U \cong \widetilde{I}$$

Consequently, $\mathscr{I}_Z$ is always a quasi-coherent sheaf of $\mathscr{O}_X$-modules. If X is a Noetherian scheme, $\mathscr{I}_Z$ is a coherent sheaf.

Proposition 10 Pullback of Quotient Sheaves.

Let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces and let $\mathcal{J}$ be a sheaf of ideals of $\mathscr{O}_Y$. For every $\mathscr{O}_X$-module $\mathscr{F}$, we have a canonical isomorphism:

$$\mathscr{F} \otimes_{\mathscr{O}_X} f^*(\mathscr{O}_Y/\mathcal{J}) \cong \mathscr{F}/(\mathcal{J}\mathscr{O}_X)\mathscr{F}.$$

Proof. Method 1: Stalk-wise Verification . It suffices to check the isomorphism on stalks. Let $p \in X$ and $q = f(p)$. The stalk of the left-hand side is:

$$\begin{aligned}(\mathscr{F} \otimes_{\mathscr{O}_X} f^*(\mathscr{O}_Y/\mathcal{J}))_p &\cong \mathscr{F}_p \otimes_{\mathscr{O}_{X,p}} (f^*(\mathscr{O}_Y/\mathcal{J}))_p \\ &\cong \mathscr{F}_p \otimes_{\mathscr{O}_{X,p}} \left((\mathscr{O}_{Y,q}/\mathcal{J}_q) \otimes_{\mathscr{O}_{Y,q}} \mathscr{O}_{X,p}\right) \\ &\cong \mathscr{F}_p \otimes_{\mathscr{O}_{X,p}} \left(\mathscr{O}_{X,p}/\mathcal{J}_q\mathscr{O}_{X,p}\right).\end{aligned}$$

Here $\mathcal{J}_q\mathscr{O}_{X,p}$ denotes the ideal in the local ring $\mathscr{O}_{X,p}$ generated by the image of $\mathcal{J}_q$. This is precisely the stalk of the inverse image ideal sheaf $(\mathcal{J}\mathscr{O}_X)_p$. By basic commutative algebra, for any ring A, module M, and ideal I, we have $M \otimes_A (A/I) \cong M/IM$. Thus:

$$\mathscr{F}_p \otimes_{\mathscr{O}_{X,p}} (\mathscr{O}_{X,p}/(\mathcal{J}\mathscr{O}_X)_p) \cong \mathscr{F}_p/(\mathcal{J}\mathscr{O}_X)_p\mathscr{F}_p.$$

This is exactly the stalk of the sheaf $\mathscr{F}/(\mathcal{J}\mathscr{O}_X)\mathscr{F}$.

Method 2: Right Exactness . Consider the short exact sequence of $\mathscr{O}_Y$-modules:

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathscr{O}_Y \longrightarrow \mathscr{O}_Y/\mathcal{J} \longrightarrow 0.$$

Apply the inverse image functor f^*. Since f^* is a left adjoint (to f_*), it is **right exact**. Thus we have an exact sequence on X:

$$f^*\mathcal{J} \longrightarrow f^*\mathcal{O}_Y \longrightarrow f^*(\mathcal{O}_Y/\mathcal{J}) \longrightarrow 0.$$

Identifying $f^*\mathcal{O}_Y$ with $\mathcal{O}_X$, the image of the first map is by definition the ideal sheaf $\mathcal{J}\mathcal{O}_X$. Thus, the cokernel is:

$$\mathcal{O}_X/(\mathcal{J}\mathcal{O}_X) \cong f^*(\mathcal{O}_Y/\mathcal{J}).$$

Now tensor this isomorphism with $\mathscr{F}$. Using the right exactness of tensor product and the algebraic identity $M \otimes_A (A/I) \cong M/IM$, we obtain:

$$\mathscr{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{O}_Y/\mathcal{J}) \cong \mathscr{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{J}\mathcal{O}_X) \cong \mathscr{F}/(\mathcal{J}\mathcal{O}_X)\mathscr{F}.$$

□

Intuition 11 Intuition behind Support of a Sheaf.

In calculus, we are familiar with the concept of the support of a function, which characterizes the set of points where the function is non-zero. Analogously, in sheaf theory, we introduce the support of a sheaf, which consists of the points where the stalks are non-trivial (i.e., non-zero).

Before constructing closed subschemes from ideal sheaves, we must understand the geometric relationship between a sheaf and the ideal that annihilates it.

Definition 12 Support of a Sheaf.

Let X be a scheme and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. The **support** of $\mathscr{F}$, denoted $\operatorname{supp}(\mathscr{F})$, is the set of points where the sheaf is not zero:

$$\operatorname{supp}(\mathscr{F}) := \{x \in \mathrm{X} \mid \mathscr{F}_x \neq 0\}.$$

Lemma 13 Algebraic Description of Support.

Let $X = \operatorname{Spec} A$ and let $\mathscr{F} = \widetilde{\mathrm{M}}$ be the quasi-coherent sheaf associated to an A-module M.

1. Generally, $x \in \operatorname{supp}(\mathscr{F}) \iff M_{\mathfrak{p}_x} \neq 0$.
2. If M is finitely generated, then the support is exactly the closed set defined by the annihilator of M:
$$\operatorname{supp}(\mathscr{F}) = V(\operatorname{Ann}_A(M)).$$
In particular, for a coherent sheaf on a Noetherian scheme, the support is always a closed subset.

Proof.

1. Definition.
2. Let $\mathfrak{p} \in \operatorname{Spec} A$. Since M is finitely generated, $M_{\mathfrak{p}} \neq 0$ is equivalent to $\operatorname{Ann}_A(M) \subseteq \mathfrak{p}$. Thus the support is the set of primes containing the annihilator, which is

$V(\mathrm{Ann}_A(M))$.

□

The following proposition establishes the geometric version of Hilbert's Nullstellensatz for schemes. It states that if a sheaf "lives" inside a closed set defined by an ideal, then algebraically it is killed by that ideal (up to a power).

Proposition 14 Schematic Nullstellensatz.

Let X be a Noetherian scheme. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module (so it is of finite type) and let $\mathscr{I}$ be a coherent sheaf of ideals of $\mathscr{O}_X$ Suppose that the support of $\mathscr{F}$ is contained in the closed subset defined by $\mathscr{I}$:

$$\mathrm{supp}(\mathscr{F}) \subseteq \mathrm{supp}(\mathscr{O}_X/\mathscr{I}) = V(\mathscr{I}).$$

Then there exists a natural number n such that $\mathscr{I}$ nilpotently annihilates $\mathscr{F}$:

$$\mathscr{I}^n\mathscr{F} = 0.$$

Proof. Since the vanishing of a sheaf is a local property and X is quasi-compact (being Noetherian), it suffices to find such an n locally and take the maximum. Cover X by finitely many affine open subschemes $U_i = \mathrm{Spec} A_i$. On each U_i, we have the correspondences:

i. $\mathscr{F}|_{U_i} \cong \widetilde{M_i}$ for a finitely generated A_i-module M_i.

ii. $\mathscr{I}|_{U_i} \cong \widetilde{I_i}$ for an ideal $I_i \subseteq A_i$.

The geometric condition $\mathrm{supp}(\mathscr{F}) \cap U_i \subseteq V(\mathscr{I}) \cap U_i$ translates via 13 above to:

$$V(\mathrm{Ann}_{A_i}(M_i)) \subseteq V(I_i).$$

In commutative algebra, $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ implies $\mathfrak{b} \subseteq \sqrt{\mathfrak{a}}$. Applying this here:

$$I_i \subseteq \sqrt{\mathrm{Ann}_{A_i}(M_i)}.$$

Since A_i is Noetherian, the ideal I_i is finitely generated, say by $f_1, \ldots, f_k$. For each generator f_j, there is a power m_j such that $f_j^{m_j} \in \mathrm{Ann}_{A_i}(M_i)$. Taking n_i large enough (e.g., sum of m_j's), we get:

$$I_i^{n_i} \subseteq \mathrm{Ann}_{A_i}(M_i) \implies I_i^{n_i} M_i = 0.$$

Since the cover is finite, let $n = \max_i\{n_i\}$. Then $\mathscr{I}^n\mathscr{F} = 0$ holds globally. □

Theorem 15 Closed Subschemes and Ideal Sheaves.

Let X be a scheme. There is an order-reversing bijection (anti-isomorphism of posets) between:

1. Let $i : Z \to X$ be a closed immersion. Then its ideal sheaf $\mathscr{I} = \ker(i^\#)$ is a quasi-coherent sheaf of ideals on X.

2. Conversely, let $\mathscr{I}$ be a quasi-coherent sheaf of ideals on X. Then the ringed space
$$Z = \Big(\operatorname{supp}(\mathscr{O}_X/\mathscr{I}), \quad (\mathscr{O}_X/\mathscr{I})|_{\operatorname{supp}(\mathscr{O}_X/\mathscr{I})}\Big)$$
is a scheme, and the natural inclusion defines a closed immersion $i : Z \to X$ with ideal sheaf $\mathscr{I}$.
3. Any two closed immersions with the same ideal sheaf are isomorphic.

Proof. (1) Let $i : Z \to X$ be a closed immersion. By definition, i is an affine morphism (locally $\operatorname{Spec}(A/J) \to \operatorname{Spec} A$). By our theorem on affine morphisms, $i_*\mathscr{O}_Z$ is a quasi-coherent $\mathscr{O}_X$-module. Since $\mathscr{O}_X$ is quasi-coherent, the kernel of the map $\mathscr{O}_X \to i_*\mathscr{O}_Z$, which is $\mathscr{I}$, is quasi-coherent.

(2) Let $\mathscr{I}$ be a quasi-coherent ideal sheaf. We construct Z in two steps: first using the relative spectrum to ensure it is a scheme, and then analyzing its structure.

Step 2a: Construction via Relative Spectrum. Consider the quotient sheaf of algebras $\mathscr{A} = \mathscr{O}_X/\mathscr{I}$. Since $\mathscr{I}$ is quasi-coherent, $\mathscr{A}$ is a quasi-coherent $\mathscr{O}_X$-algebra. We define the scheme:
$$Z := \mathsf{Spec}_X(\mathscr{A}).$$
Let $i : Z \to X$ be the structure morphism. By the construction of the relative spectrum, we canonically have $i_*\mathscr{O}_Z \cong \mathscr{A} = \mathscr{O}_X/\mathscr{I}$.

Step 2b: Unpacking the Topology (The Support). We analyze Z locally. Let $U = \operatorname{Spec} A$ be an affine open of X. Then $\mathscr{I}|_U = \widetilde{I}$ for an ideal $I \subset A$. The restriction of the relative spectrum is:
$$Z|_U \cong \operatorname{Spec}(A/I).$$
The morphism $i|_U : \operatorname{Spec}(A/I) \to \operatorname{Spec} A$ is induced by the projection $A \to A/I$. Topologically, the image of this map is the closed set $V(I) \subset \operatorname{Spec} A$. Recall from our study of supports (14) that for the sheaf $\widetilde{A/I}$, the support is exactly $V(\operatorname{Ann}(A/I)) = V(I)$. Gluing these local observations, the underlying topological space of Z is precisely the support of the sheaf $\mathscr{O}_X/\mathscr{I}$:
$$|Z| \cong \operatorname{supp}(\mathscr{O}_X/\mathscr{I}) \subseteq |X|.$$
Thus, i induces a homeomorphism from Z onto the closed subset $\operatorname{supp}(\mathscr{O}_X/\mathscr{I})$.

Step 2c: Unpacking the Structure Sheaf. We have established that i is a homeomorphism onto its image $K = \operatorname{supp}(\mathscr{O}_X/\mathscr{I})$. We check the sheaf on Z. By the property of the relative spectrum, $i_*\mathscr{O}_Z \cong \mathscr{O}_X/\mathscr{I}$. For any closed immersion i, the structure sheaf $\mathscr{O}_Z$ is obtained by restricting the pushforward $i_*\mathscr{O}_Z$ to the closed subset K. Explicitly:
$$\mathscr{O}_Z = i^{-1}(i_*\mathscr{O}_Z) = i^{-1}(\mathscr{O}_X/\mathscr{I}).$$
This is precisely the restriction of the sheaf $\mathscr{O}_X/\mathscr{I}$ to its support. Thus, the abstractly constructed $\mathsf{Spec}_X(\mathscr{A})$ coincides with the concrete ringed space $(\operatorname{supp}(\mathscr{O}_X/\mathscr{I}), (\mathscr{O}_X/\mathscr{I})|_{\operatorname{supp}})$.

(3) Let $i' : Z' \to X$ be another closed immersion with ideal sheaf $\mathscr{I}$. Then $i'_*\mathscr{O}_{Z'} \cong \mathscr{O}_X/\mathscr{I}$. Since Z' is a closed subscheme, i' is an affine morphism. By the Uniqueness Theorem for affine morphisms (Prop 1.4.12 equivalent), Z' is uniquely isomorphic to $\mathsf{Spec}_X(i'_*\mathscr{O}_{Z'}) \cong \mathsf{Spec}_X(\mathscr{O}_X/\mathscr{I}) = Z$. □

Note 16 Categorical Perspective.

We can view the set of closed subschemes as a category (a poset category) where morphisms are inclusions $Z \hookrightarrow Z'$. The theorem states that this category is anti-equivalent to the poset category of quasi-coherent ideal sheaves:

$$Z \subseteq Z' \iff \mathscr{I}_{Z'} \subseteq \mathscr{I}_Z.$$

Geometrically: "Larger ideal defines smaller space."

The set-theoretic image $f(X)$ of a morphism is rarely a scheme (it is often merely a constructible set). However, algebra allows us to define the "smallest closed subscheme" through which a morphism factors. This is the correct schematic generalization of the topological closure of the image.

Definition 17 Scheme-Theoretic Image.

Let $f : X \to Y$ be a morphism of schemes. The **scheme-theoretic image** of f is the closed subscheme $Z \subseteq Y$ defined by the ideal sheaf:

$$\mathscr{I} := \ker(f^\# : \mathscr{O}_Y \longrightarrow f_*\mathscr{O}_X).$$

We denote this subscheme by $\mathrm{Im}(f)$.

Note 18 Well-Definedness of Scheme-Theoretic Image.

For $\mathrm{Im}(f)$ to be a well-defined closed subscheme, its defining ideal sheaf $\mathscr{I}$ must be quasi-coherent. Recall our previous result (6): if f is quasi-compact and quasi-separated (QCQS), then $f_*\mathscr{O}_X$ is a quasi-coherent $\mathscr{O}_Y$-module. Since $\mathscr{O}_Y$ is quasi-coherent, the kernel $\mathscr{I}$ is quasi-coherent. Thus, the scheme-theoretic image is well-behaved for QCQS morphisms.

Proposition 19 Universal Property of Scheme-Theoretic Image.

Let $f : X \to Y$ be a QCQS morphism. Let $Z = \mathrm{Im}(f)$ with closed immersion $i : Z \hookrightarrow Y$.

1. **Factorization:** The morphism f factors uniquely through Z. That is, there exists a unique morphism $g : X \to Z$ such that $f = i \circ g$.
2. **Minimality:** If $Z' \hookrightarrow Y$ is any other closed subscheme such that f factors through Z', then $Z \subseteq Z'$ (as subschemes, meaning $\mathscr{I}_{Z'} \subseteq \mathscr{I}_Z$).
3. **Topology:** The underlying topological space of Z is exactly the closure of the set-theoretic image, $\overline{f(X)}$.

Proof. (1) By definition, $\mathscr{I} = \ker(\mathscr{O}_Y \to f_*\mathscr{O}_X)$. By the universal property of the kernel/quotient, the map $f^\#$ factors through the quotient sheaf:

$$\mathscr{O}_Y \xrightarrow{\text{canonical}} \mathscr{O}_Y/\mathscr{I} \xrightarrow{\bar{f}^\#} f_*\mathscr{O}_X.$$

Recall that the structure sheaf of Z is $\mathscr{O}_Z = i^{-1}(\mathscr{O}_Y/\mathscr{I})$. By the adjunction of (f^*, f_*)

and (i^*, i_*), maps $\mathscr{O}_Y/\mathscr{I} \to f_*\mathscr{O}_X$ correspond to maps $f^*(\mathscr{O}_Y/\mathscr{I}) \to \mathscr{O}_X$. Since f factors through the support of $\mathscr{O}_Y/\mathscr{I}$, this defines the morphism $g : X \to Z$.

(2) Suppose f factors through $j : Z' \hookrightarrow Y$ via $h : X \to Z'$. Then $f^\#$ factors as:

$$\mathscr{O}_Y \xrightarrow{j^\#} j_*\mathscr{O}_{Z'} \to j_*h_*\mathscr{O}_X = f_*\mathscr{O}_X.$$

The kernel of the composition $(f^\#)$ must contain the kernel of the first map $(j^\#)$. Thus $\mathscr{I}_{Z'} = \ker(j^\#) \subseteq \ker(f^\#) = \mathscr{I}_Z$. Geometrically, a smaller ideal defines a larger subscheme, so $Z \subseteq Z'$.

(3) We verify this locally. Let $Y = \operatorname{Spec} A$. Then f maps to Y via a ring map $\phi : A \to \Gamma(X, \mathscr{O}_X)$. The ideal sheaf $\mathscr{I}$ corresponds to the kernel $I = \ker(\phi)$. The scheme Z corresponds to $\operatorname{Spec}(A/I)$. The map on rings factors as $A \twoheadrightarrow A/I \hookrightarrow \Gamma(X, \mathscr{O}_X)$. The image of the map of spectra $\operatorname{Spec}(\Gamma(X, \mathscr{O}_X)) \to \operatorname{Spec} A$ is dense in $V(I)$. Since the map $X \to \operatorname{Spec}(\Gamma(X, \mathscr{O}_X))$ is dominant (roughly), the image $f(X)$ is dense in Z. More precisely, $\operatorname{supp}(f_*\mathscr{O}_X) = \overline{f(X)}$ is a general fact about the support of pushforwards. And we established in the previous section that the underlying space of Z is $\operatorname{supp}(\mathscr{O}_Y/\mathscr{I})$. Since $\mathscr{O}_Y/\mathscr{I} \hookrightarrow f_*\mathscr{O}_X$, their supports are essentially the same (closure). □

This allows us to cleanly dissect the structure of immersions.

Corollary 20 Factorization of Immersions.

Let $f : W \to Y$ be an immersion quasi-compact, then f factors uniquely as:

$$W \xrightarrow{\text{open } j} Z \xrightarrow{\text{closed } i} Y$$

where $Z = \operatorname{Im}(f)$ is the scheme-theoretic image (the closure of W equipped with the reduced induced structure).

Proof. Let Z be the scheme-theoretic image of f. By the universal property, f factors as $W \xrightarrow{j} Z \xrightarrow{i} Y$. Since i is a closed immersion, it suffices to show j is an open immersion. Since f is an immersion, topologically W is homeomorphic to an open subset U of a closed subset K of Y. The topological space of Z is $\overline{f(W)} = \overline{U}$, which is the closure of W. Thus, W is an open subset of Z. Algebraically, we check stalks. For $w \in W$, the map $\mathscr{O}_{Y,f(w)} \to \mathscr{O}_{W,w}$ is surjective. It factors through $\mathscr{O}_{Z,f(w)}$. Thus the map $\mathscr{O}_{Z,f(w)} \to \mathscr{O}_{W,w}$ is surjective. Since $\mathscr{O}_Z$ is a subsheaf of $f_*\mathscr{O}_W$, it is injective. Hence j is an isomorphism on stalks, making it an open immersion. □

6.3 How to Check a Scheme is Affine?

Proposition 21 Affine Criterion by Generating Sections.

A scheme $(X, \mathscr{O}_X)$ is an affine scheme if and only if there exist finitely many global sections $f_1, \dots, f_n \in \Gamma(X, \mathscr{O}_X)$ such that:

(i) The sections generate the unit ideal in the global section ring $A = \Gamma(X, \mathscr{O}_X)$ (i.e.,

$(f_1, \ldots, f_n) = A$).

(ii) For each i, the open subscheme $X_{f_i} = \{x \in X \mid (f_i)_x \notin \mathfrak{m}_x\}$ is affine.

Proof. The "only if" direction is trivial: if $X = \operatorname{Spec} A$, take $f_1 = 1$. Then $X_1 = X$ is affine.

We focus on the "if" direction. Let $A = \Gamma(X, \mathscr{O}_X)$. We proceed in three steps.

Step 1: Construction of the Canonical Morphism. By the adjunction between the functor of points and the global section functor, the identity homomorphism $\mathrm{id} : A \to \Gamma(X, \mathscr{O}_X)$ induces a canonical morphism of schemes:

$$\psi : X \longrightarrow \operatorname{Spec} A.$$

We aim to show that ψ is an isomorphism.

Step 2: Compatibility of Open Covers. Consider the basic open sets $D(f_i) \subseteq \operatorname{Spec} A$. Since the elements $f_1, \ldots, f_n$ generate the unit ideal of A, the union of these open sets covers the target:

$$\operatorname{Spec} A = \bigcup_{i=1}^{n} D(f_i).$$

Now we examine the inverse image $\psi^{-1}(D(f_i))$. By the definition of the canonical morphism, a point $x \in X$ maps to a prime ideal $\mathfrak{p} \in D(f_i)$ if and only if $f_i \notin \mathfrak{p}$, which geometrically means the germ of f_i at x is not in the maximal ideal $\mathfrak{m}_x$. This is precisely the definition of the open set X_{f_i}. Thus:

$$\psi^{-1}(D(f_i)) = X_{f_i}.$$

Consequently, the collection $\{X_{f_i}\}$ forms an open covering of X.

Step 3: Local Isomorphisms. It suffices to show that the restriction $\psi|_{X_{f_i}} : X_{f_i} \to D(f_i)$ is an isomorphism for each i. The target $D(f_i)$ is naturally isomorphic to the affine scheme $\operatorname{Spec}(A_{f_i})$. By hypothesis, the source X_{f_i} is an affine scheme. Let $X_{f_i} \cong \operatorname{Spec} B_i$.

To prove $\operatorname{Spec} B_i \cong \operatorname{Spec} A_{f_i}$, we need to compare their rings. The restriction of the map ψ corresponds to the ring homomorphism $A_{f_i} \to B_i$ induced by restriction of sections $A \to \Gamma(X_{f_i}, \mathscr{O}_X)$. We invoke a key lemma on the localization of global sections:

Lemma: If a scheme X admits a finite cover by affine opens U_i such that each intersection $U_i \cap U_j$ is affine (or covered by affines), then for any $f \in \Gamma(X, \mathscr{O}_X)$, the natural map $(\Gamma(X, \mathscr{O}_X))_f \to \Gamma(X_f, \mathscr{O}_X)$ is an isomorphism.

Let's verify the condition for our cover $\{X_{f_i}\}$:

- Each X_{f_i} is affine ($\operatorname{Spec} B_i$).
- The intersection $X_{f_i} \cap X_{f_j} = (X_{f_i})_{f_j}$. Since $X_{f_i} = \operatorname{Spec} B_i$, this open set corresponds to the distinguished open $D(f_j|_{X_{f_i}}) \subseteq \operatorname{Spec} B_i$.
- A distinguished open subset of an affine scheme is affine ($\operatorname{Spec}(B_i)_{f_j}$).

Thus, the intersection of any two sets in our cover is affine. The condition of the lemma is satisfied.

Therefore, we have an isomorphism of rings:

$$A_{f_i} \xrightarrow{\sim} \Gamma(X_{f_i}, \mathscr{O}_X) = B_i.$$

This implies that $\psi|_{X_{f_i}} : X_{f_i} \to D(f_i)$ is an isomorphism of schemes.

Since ψ is a local isomorphism on a covering of X that maps surjectively to a covering of $\operatorname{Spec} A$, ψ is a global isomorphism. Hence $X \cong \operatorname{Spec} A$ is affine. □

Theorem 22 Serre's Criterion.

Let X be a Noetherian scheme. Then the following conditions are equivalent:

(i) X is an affine scheme.

(ii) $H^i(X, \mathscr{F}) = 0$ for all quasi-coherent sheaves $\mathscr{F}$ and all $i > 0$.

(iii) $H^1(X, \mathscr{I}) = 0$ for all coherent sheaves of ideals $\mathscr{I}$.

Proof. The implication (i) ⇒ (ii) is simply **Serre's Vanishing Theorem** (Theorem B) for affine schemes, which we have already established.

The implication (ii) ⇒ (iii) is trivial because every coherent sheaf of ideals is, in particular, a quasi-coherent sheaf, and the condition holds for $i = 1$.

The core of the proof lies in the implication (iii) ⇒ (i). We assume that $H^1(X, \mathscr{I}) = 0$ for every coherent sheaf of ideals $\mathscr{I}$, and we must prove X is affine. We will use the **Affine Criterion by Generating Sections** (**Proposition 21**). To do this, we need to construct a set of global sections $f_1, \ldots, f_r \in \Gamma(X, \mathscr{O}_X)$ such that:

(A) Each principal open set X_{f_i} is affine.

(B) The sections $f_1, \ldots, f_r$ generate the unit ideal in $\Gamma(X, \mathscr{O}_X)$.

Step 1: Constructing Affine Neighborhoods for Closed Points. We claim that every closed point $P \in X$ has an affine open neighborhood of the form X_f for some $f \in \Gamma(X, \mathscr{O}_X)$.

Let P be a closed point of X. Since X is a scheme, there exists an affine open neighborhood $U = \operatorname{Spec} A$ containing P. Let $Y = X \setminus U$. Since U is open, Y is a closed subset of X. Consider the union $Y \cup \{P\}$. Since X is Noetherian (or simply T_1), the point $\{P\}$ is closed, so $Z = Y \cup \{P\}$ is a closed subset of X.

We define two ideal sheaves with the reduced induced scheme structure:

- $\mathscr{I}_Y$: the ideal sheaf of the closed set Y.
- $\mathscr{I}_Z$: the ideal sheaf of the closed set $Z = Y \cup \{P\}$.

Clearly $Y \subset Z$, so we have an inclusion of ideals $\mathscr{I}_Z \subseteq \mathscr{I}_Y$. Consider the quotient sheaf $\mathcal{Q} = \mathscr{I}_Y / \mathscr{I}_Z$. A section is in $\mathscr{I}_Y$ if it vanishes on Y. It is in $\mathscr{I}_Z$ if it vanishes on Y and at P.

Thus, the quotient is supported exactly at P. Specifically, we have a short exact sequence of coherent sheaves:

$$0 \longrightarrow \mathscr{I}_Z \longrightarrow \mathscr{I}_Y \longrightarrow k(P) \longrightarrow 0,$$

where $k(P)$ is the skyscraper sheaf at P with value the residue field $\kappa(P)$.

Now, apply the long exact sequence of cohomology. The relevant segment is:

$$\Gamma(X, \mathscr{I}_Y) \longrightarrow \Gamma(X, k(P)) \longrightarrow H^1(X, \mathscr{I}_Z).$$

By our hypothesis (iii), $H^1(X, \mathscr{I}_Z) = 0$. Therefore, the map $\Gamma(X, \mathscr{I}_Y) \to \Gamma(X, k(P))$ is **surjective**.

The group $\Gamma(X, k(P))$ is simply the field $\kappa(P)$. The element $1 \in \kappa(P)$ must lift to a global section $f \in \Gamma(X, \mathscr{I}_Y)$. Let's analyze the properties of this section f:

- Since $f \in \Gamma(X, \mathscr{I}_Y)$, f vanishes on $Y = X \setminus U$. This implies that the open set where f does not vanish, X_f, is contained in U ($X_f \subseteq U$).
- Since f maps to 1 in $k(P)$, $f(P) \neq 0$. This implies $P \in X_f$.

Now, consider the structure of X_f. Since $X_f \subseteq U$ and $U = \operatorname{Spec} A$ is affine, X_f is exactly the distinguished open set $D(f|_U)$ inside the affine scheme U. Thus, X_f is itself affine (isomorphic to $\operatorname{Spec} A_f$).

Step 2: Covering X by Basic Affine Opens. We have shown that for every closed point P, there is an element $f \in \Gamma(X, \mathscr{O}_X)$ such that $P \in X_f$ and X_f is affine. The union of such open sets $\bigcup X_f$ contains all closed points of X. In a Noetherian scheme, the set of closed points is dense; in fact, any open set containing all closed points must be the whole space X (unless X is empty). Since X is Noetherian, it is quasi-compact. Thus, we can select a **finite** number of global sections $f_1, \ldots, f_r \in \Gamma(X, \mathscr{O}_X)$ such that:

$$X = \bigcup_{i=1}^{r} X_{f_i},$$

and each X_{f_i} is affine. This satisfies condition (A) of the Affine Criterion.

Step 3: Generating the Unit Ideal. It remains to prove condition (B): that the ideal generated by $f_1, \ldots, f_r$ in $A = \Gamma(X, \mathscr{O}_X)$ is the unit ideal. Consider the sheaf morphism defined by these sections:

$$\alpha : \mathscr{O}_X^{\oplus r} \longrightarrow \mathscr{O}_X, \quad (s_1, \ldots, s_r) \longmapsto \sum_{i=1}^{r} s_i f_i.$$

Since $X = \bigcup X_{f_i}$, for any point $x \in X$, there is at least one f_i such that $(f_i)_x$ is a unit in the local ring $\mathscr{O}_{X,x}$. Thus, the map α is surjective as a morphism of sheaves. Let $\mathscr{F} = \operatorname{Ker}(\alpha)$. We have a short exact sequence:

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{O}_X^{\oplus r} \xrightarrow{\alpha} \mathscr{O}_X \longrightarrow 0.$$

Taking global sections yields:

$$\Gamma(X, \mathscr{O}_X^{\oplus r}) \xrightarrow{\alpha_X} \Gamma(X, \mathscr{O}_X) \longrightarrow H^1(X, \mathscr{F}).$$

The map α_X sends (s_i) to $\sum s_i f_i$. Its image is the ideal $(f_1, \dots, f_r)$. To show this ideal is A (i.e., contains 1), it suffices to show α_X is surjective. This happens if and only if $H^1(X, \mathscr{F}) = 0$.

The kernel $\mathscr{F}$ is a coherent sheaf (being the kernel of a map between coherent sheaves on a Noetherian scheme), but it is not necessarily an ideal sheaf. However, we can use a **filtration argument** to relate it to ideal sheaves.

We filter the free sheaf $\mathscr{E} = \mathscr{O}_X^{\oplus r}$ by subsheaves $\mathscr{E}_k = \mathscr{O}_X^{\oplus k} \oplus 0^{\oplus(r-k)}$ (the sum of the first k factors). This induces a filtration on the subsheaf $\mathscr{F}$:

$$\mathscr{F}_k = \mathscr{F} \cap \mathscr{E}_k.$$

We have $\mathscr{F}_0 = 0$ and $\mathscr{F}_r = \mathscr{F}$. Consider the successive quotients:

$$\mathscr{F}_k / \mathscr{F}_{k-1} \cong \frac{\mathscr{F} \cap \mathscr{E}_k}{\mathscr{F} \cap \mathscr{E}_{k-1}}.$$

Since $\mathscr{E}_k / \mathscr{E}_{k-1} \cong \mathscr{O}_X$, the quotient $\mathscr{F}_k / \mathscr{F}_{k-1}$ is isomorphic to a subsheaf of $\mathscr{O}_X$. In other words, each quotient is a **coherent ideal sheaf** (or zero). Let's call them $\mathcal{J}_k$.

We have short exact sequences:

$$0 \longrightarrow \mathscr{F}_{k-1} \longrightarrow \mathscr{F}_k \longrightarrow \mathcal{J}_k \longrightarrow 0.$$

The long exact sequence of cohomology gives:

$$\cdots \longrightarrow H^1(X, \mathscr{F}_{k-1}) \longrightarrow H^1(X, \mathscr{F}_k) \longrightarrow H^1(X, \mathcal{J}_k) \longrightarrow \dots$$

By hypothesis (iii), $H^1(X, \mathcal{J}_k) = 0$ for all ideal sheaves. For $k = 1$, $\mathscr{F}_1$ is an ideal sheaf, so $H^1(X, \mathscr{F}_1) = 0$. By induction, if $H^1(X, \mathscr{F}_{k-1}) = 0$, then $H^1(X, \mathscr{F}_k) = 0$. Thus, $H^1(X, \mathscr{F}) = H^1(X, \mathscr{F}_r) = 0$.

Since $H^1(X, \mathscr{F}) = 0$, the global section map α_X is surjective. Thus $1 \in \Gamma(X, \mathscr{O}_X)$ lies in the ideal generated by $f_1, \dots, f_r$.

Conclusion: We have found $f_1, \dots, f_r$ such that they generate the unit ideal and each X_{f_i} is affine. By the **Affine Criterion** (**Proposition 21**), X is an affine scheme. □

6.4 The Relative Spectrum

Intuition 23 Relative Spectrum Intuition.

Following our previous introduction of the concept of relative schemes, we now introduce the concept of the Relative Spectrum. The key significance of this concept lies in establishing an anti-equivalence between the category of quasi-coherent algebras (over a base scheme S) and the category of affine morphisms (into S). We will find that any affine morphism $f : X \to S$ in fact corresponds to a quasi-coherent algebra $\mathcal{A}$ on S, and vice versa. This correspondence allows us to describe the pushforward functor (or direct image functor) more clearly and precisely.

In classical algebraic geometry, varieties are defined over a fixed field k. In scheme theory, we adopt the *relative point of view*: we study schemes X over a base scheme S. The most fundamental relative construction is the "relative spectrum," which generalizes the construction of $\operatorname{Spec} A$ from a ring A.

This section unifies the construction of affine morphisms, their universal properties, and the equivalence of categories into a single coherent framework.

Definition 24 Relative Spectrum.

Let S be a scheme and let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. There exists a scheme over S, denoted by

$$\pi : \mathsf{Spec}_S(\mathcal{A}) \longrightarrow S,$$

unique up to unique isomorphism, constructed as follows: For any affine open subset $U = \operatorname{Spec} R \subseteq S$, the restriction $\mathcal{A}|_U$ corresponds to an R-algebra $A = \Gamma(U, \mathcal{A})$. We define the preimage $\pi^{-1}(U)$ to be the affine scheme $\operatorname{Spec} A$. These local affine schemes glue together along overlaps (verified via the localization property of structure sheaves) to form the global scheme $\mathsf{Spec}_S(\mathcal{A})$.

Theorem 25 Existence and Uniqueness of Relative Spectrum.

Let S be a scheme and let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. There exists an S-scheme $f : X \to S$, unique up to unique isomorphism, such that for every affine open subset $U = \operatorname{Spec} R \subseteq S$, the preimage $f^{-1}(U)$ is canonically isomorphic to $\operatorname{Spec} \mathcal{A}(U)$ as a scheme over U.

We denote this scheme by $X = \mathsf{Spec}_S(\mathcal{A})$.

Proof. The proof proceeds in two steps: constructing local models and then gluing them together.

Step 1: Local Construction. Let $U \subseteq S$ be *any* affine open subscheme, $U \cong \operatorname{Spec} R$. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_S$-algebra.

Restricting $\mathcal{A}$ to U, we obtain an $\mathcal{O}_U$-algebra $\mathcal{A}|_U$. Since $\mathcal{A}$ is quasi-coherent, $\mathcal{A}|_U$ is the sheaf associated to the R-algebra $A_U := \Gamma(U, \mathcal{A})$.

We define the *canonical local scheme* X_U over U as the affine spectrum of this algebra:

$$X_U := \operatorname{Spec} A_U$$

The structure homomorphism $\phi_U : R \to A_U$ induces a canonical affine morphism of schemes $f_U : X_U \to U$.

Step 2: Gluing Data. To construct the global scheme X, we must gather the local data and glue them over their overlaps.

Let $\{U_i\}_{i \in I}$ be an affine open covering of S, $U_i \cong \operatorname{Spec} R_i$. According to Step 1, we have constructed local schemes $X_i := X_{U_i}$ with structure morphisms $f_i : X_i \to U_i$.

Let $U_{ij} = U_i \cap U_j$. The inverse images $f_i^{-1}(U_{ij}) \subseteq X_i$ and $f_j^{-1}(U_{ij}) \subseteq X_j$ are the regions to be glued.

Since f_i and f_j are affine morphisms, the restrictions $f_i^{-1}(U_{ij}) \to U_{ij}$ and $f_j^{-1}(U_{ij}) \to U_{ij}$ are also affine morphisms, defined by the restriction of the $\mathcal{O}_S$-algebra $\mathcal{A}$ to U_{ij}.

Specifically, the set of schemes $\{f_i^{-1}(U_{ij}) \longrightarrow U_{ij}\}_{i\in I}$ is canonically isomorphic to the single object defined in Step 1 for U_{ij}:

$$f_i^{-1}(U_{ij}) \cong \mathrm{Spec}_{U_{ij}}(\mathcal{A}|_{U_{ij}}) \cong f_j^{-1}(U_{ij})$$

Therefore, we have a canonical isomorphism of schemes over U_{ij}:

$$\theta_{ij} : f_i^{-1}(U_{ij}) \xrightarrow{\sim} f_j^{-1}(U_{ij}).$$

For any common affine open $V \subseteq U_{ij}$, θ_{ij} is induced by the identity map on the ring $\Gamma(V, \mathcal{A})$. These isomorphisms θ_{ij} form the necessary data to glue the topological spaces $|X_i|$ and the sheaves of rings $\mathcal{O}_{X_i}$.

Step 3: Cocycle Condition. We verify the cocycle condition on the triple intersection $U_{ijk} = U_i \cap U_j \cap U_k$. We need to show that

$$\theta_{ik} = \theta_{ij} \circ \theta_{jk}$$

on the relevant domains. Since these schemes are locally affine (over small affine opens $V \subseteq U_{ijk}$), the morphisms θ are induced by the restriction maps of the sheaf $\mathcal{A}$. Specifically, θ_{ij} is induced by the identity map of the section algebra $\mathcal{A}(V)$. The cocycle condition on the schemes thus reduces to the equality of identity maps on the rings:

$$\mathrm{id}_{\mathcal{A}(V)} = \mathrm{id}_{\mathcal{A}(V)} \circ \mathrm{id}_{\mathcal{A}(V)},$$

which is trivially satisfied.

Step 4: Conclusion. By the Gluing Lemma for schemes, there exists a global scheme X obtained by gluing the X_i via θ_{ij}. The local morphisms $f_i : X_i \to U_i$ are compatible with the gluing, yielding a global morphism $f : X \to S$. By construction, for any affine U_i in the cover, $f^{-1}(U_i) \cong X_i = \mathrm{Spec}\,\mathcal{A}(U_i)$. For an *arbitrary* affine open $V \subseteq S$, one can cover V by standard basic opens of the U_i's and use the sheaf property to show $f^{-1}(V) \cong \mathrm{Spec}\,\mathcal{A}(V)$.

Uniqueness: Suppose (X', f') is another such scheme. Covering S by affines U_i, we have unique isomorphisms $\psi_i : f'^{-1}(U_i) \xrightarrow{\sim} \mathrm{Spec}\,\mathcal{A}(U_i) = f^{-1}(U_i)$. Since both ψ_i and the gluing maps are determined canonically by the sheaf $\mathcal{A}$, the ψ_i satisfy the compatibility condition $\psi_i = \theta_{ij} \circ \psi_j$ on overlaps. Thus, the ψ_i glue to a unique global isomorphism $\psi : X' \to X$. □

Having constructed the relative spectrum $\mathsf{Spec}_S(\mathcal{B})$, we now establish its defining universal property: it represents the functor of "morphisms into an affine scheme over S".

Theorem 26 Universal Property of Relative Spectrum.

Let S be a scheme and $\mathcal{B}$ a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Let $f : X \to S$ be any S-scheme. There is a canonical bijection:

$$\Phi : \mathrm{Hom}_{\mathsf{Sch}/S}(X, \mathsf{Spec}_S\mathcal{B}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_S\text{-alg}}(\mathcal{B}, f_*\mathcal{O}_X).$$

Proof. Let $Y = \mathsf{Spec}_S \mathcal{B}$ and let $\pi : Y \to S$ be the structure morphism. Note that by construction, $\pi_* \mathcal{O}_Y \cong \mathcal{B}$ canonically.

Step 1: The Map Φ (Global to Algebraic). Given an S-morphism $h : X \to Y$ (so $\pi \circ h = f$), we have the induced homomorphism of structure sheaves $h^\sharp : \mathcal{O}_Y \to h_* \mathcal{O}_X$. Applying π_* to this map, we obtain:

$$\pi_*(h^\sharp) : \pi_* \mathcal{O}_Y \longrightarrow \pi_* h_* \mathcal{O}_X.$$

Since $\pi_* \mathcal{O}_Y \cong \mathcal{B}$ and $\pi_* h_* = (\pi \circ h)_* = f_*$, this gives a homomorphism of $\mathcal{O}_S$-algebras:

$$\psi : \mathcal{B} \longrightarrow f_* \mathcal{O}_X.$$

We define $\Phi(h) = \psi$.

Step 2: The Inverse Map Ψ (Algebraic to Global). Suppose we are given a homomorphism of $\mathcal{O}_S$-algebras $\psi : \mathcal{B} \to f_* \mathcal{O}_X$. We construct a morphism $h : X \to Y$ locally and glue.

Cover S by affine open subsets $U_i = \operatorname{Spec} R_i$. Let $\mathcal{B}|_{U_i}$ correspond to the R_i-algebra $B_i = \Gamma(U_i, \mathcal{B})$. By the construction of the relative spectrum, $\pi^{-1}(U_i) \cong \operatorname{Spec} B_i$. Restricting ψ to U_i, we get a homomorphism of sheaves on U_i:

$$\psi|_{U_i} : \mathcal{B}|_{U_i} \longrightarrow (f_* \mathcal{O}_X)|_{U_i}.$$

Taking global sections on U_i, this corresponds to a ring homomorphism:

$$\psi_i : B_i \longrightarrow \Gamma(U_i, f_* \mathcal{O}_X) = \Gamma(f^{-1}(U_i), \mathcal{O}_X).$$

Let $V_i = f^{-1}(U_i)$. The map ψ_i is a homomorphism from B_i to the global sections of the scheme V_i. By the **canonical adjunction of affine schemes** (the classical property of Spec), such a ring homomorphism corresponds to a unique morphism of schemes:

$$h_i : V_i \longrightarrow \operatorname{Spec} B_i = \pi^{-1}(U_i).$$

Moreover, this morphism commutes with the maps to $U_i = \operatorname{Spec} R_i$, so h_i is a morphism over U_i.

Step 3: Gluing. To show that $\{h_i\}$ glue to a global morphism $h : X \to Y$, we observe that for any affine open $W \subseteq U_i \cap U_j$, the morphism $h_i|_W$ is determined uniquely by the map of rings $\mathcal{B}(W) \to \Gamma(f^{-1}(W), \mathcal{O}_X)$ induced by ψ. Since ψ is a globally defined sheaf morphism, the ring maps induced on overlaps are identical. Thus, h_i and h_j agree on $f^{-1}(U_i \cap U_j)$. Therefore, there exists a unique global morphism $h : X \to Y$ inducing ψ.

Φ and Ψ are inverse to each other. □

Note 27 Distinguishing Two Adjunctions Involving the Direct Image Functor.

The reader may recall the familiar adjunction (f^*, f_*) between modules. It is crucial to distinguish the two different roles played by the direct image functor f_* in algebraic geometry:

1. **Module Adjunction (Fixed Geometry):** Fix a morphism $f : X \to S$. The functor $f_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(S)$ moves *modules* between fixed spaces. Its left adjoint is the

inverse image f^*.

$$\mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{G}, f_*\mathscr{F}).$$

2. **Relative Spectrum Adjunction (Varying Geometry):** Fix the base S. The functor $\mathcal{R} : (\mathsf{Sch}/S)^{op} \longrightarrow \mathsf{Alg}(\mathscr{O}_S)$ given by $(X \xrightarrow{f} S) \mapsto f_*\mathscr{O}_X$ relates *spaces* to algebras. Its adjoint is the relative spectrum Spec_S.

$$\mathrm{Hom}_S(X, \mathsf{Spec}_S\mathcal{B}) \cong \mathrm{Hom}_{\mathscr{O}_S\text{-alg}}(\mathcal{B}, f_*\mathscr{O}_X).$$

In the first case, f_* acts on the category of modules; in the second, it acts (contravariantly) on the category of schemes to produce the structure algebra.

With the construction and the adjunction established, we now prove that affine morphisms are precisely those schemes arising from quasi-coherent algebras.

Theorem 28 Equivalence of Categories between Quasi-Coherent Algebras and Affine Morphisms.

Let S be a scheme. The functor $\mathsf{Spec}_S(-)$ establishes an anti-equivalence of categories between:

1. The category of quasi-coherent $\mathscr{O}_S$-algebras, $\mathsf{QCohAlg}(S)$.
2. The category of affine morphisms to S, AffSch/S.

The inverse functor is given by taking the global structure sheaf:

$$(f : X \to S) \longmapsto f_*\mathscr{O}_X.$$

Specifically, for any quasi-coherent algebra $\mathcal{B}$, the canonical map $\mathcal{B} \to \pi_*\mathscr{O}_{\mathsf{Spec}_S\mathcal{B}}$ is an isomorphism. Conversely, for any affine morphism $f : X \to S$, the canonical morphism $X \to \mathsf{Spec}_S(f_*\mathscr{O}_X)$ is an isomorphism.

Proof. We check that the unit and counit of the adjunction are isomorphisms.

Step 1: From Algebra to Scheme and back. Let $\mathcal{B}$ be a quasi-coherent $\mathscr{O}_S$-algebra and let $Y = \mathsf{Spec}_S\mathcal{B}$ with structure morphism $\pi : Y \to S$. We need to show that the canonical map $\eta : \mathcal{B} \to \pi_*\mathscr{O}_Y$ is an isomorphism. This is a local question on S. Let $U = \operatorname{Spec} R \subseteq S$ be an affine open. By the construction of the relative spectrum (Step 1 of 25), the restriction $\pi^{-1}(U)$ is naturally isomorphic to $\operatorname{Spec}(\mathcal{B}(U))$. For an affine scheme $\operatorname{Spec} B$, the global sections of the structure sheaf recover the ring B. Thus:

$$(\pi_*\mathscr{O}_Y)(U) = \Gamma(\pi^{-1}(U), \mathscr{O}_Y) \cong \Gamma(\operatorname{Spec}(\mathcal{B}(U)), \mathscr{O}) \cong \mathcal{B}(U).$$

Since this holds for every affine open U, the sheaf map η is an isomorphism.

Step 2: From Affine Morphism to Algebra and back. Let $f : X \to S$ be an affine morphism. Let $\mathcal{A} = f_*\mathscr{O}_X$. By 6, the identity map $\mathrm{id} : \mathcal{A} \to f_*\mathscr{O}_X$ corresponds to a canonical S-morphism:

$$\epsilon : X \longrightarrow \mathsf{Spec}_S(f_*\mathscr{O}_X) = \mathsf{Spec}_S(\mathcal{A}).$$

We need to show ϵ is an isomorphism. Again, the question is local on S. Let $U = \operatorname{Spec} R \subseteq S$ be an affine open. Since f is an affine morphism, $f^{-1}(U)$ is an affine scheme, say $\operatorname{Spec} A$.

Then $(f_*\mathscr{O}_X)(U) = \Gamma(f^{-1}(U), \mathscr{O}_X) = \Gamma(\operatorname{Spec} A, \mathscr{O}) = A$. So over U, the relative spectrum $\mathsf{Spec}_S(\mathscr{A})$ restricts to $\operatorname{Spec} A$. The map $\epsilon|_U : f^{-1}(U) \to \mathsf{Spec}_S(\mathscr{A})|_U$ becomes the map of schemes induced by the identity map on the ring A:

$$\epsilon|_U : \operatorname{Spec} A \longrightarrow \operatorname{Spec} A,$$

which is clearly an isomorphism. Since ϵ is locally an isomorphism on a cover of S, it is a global isomorphism. □

By establishing the theory of the relative spectrum, the classical propositions regarding affine morphisms (often proved via tedious local gluing in standard texts) become immediate corollaries of our main theorems.

Corollary 29 Morphisms into Affine Schemes.

Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes. Assume g is an affine morphism. Then there is a canonical bijection:

$$\operatorname{Hom}_S(X, Y) \cong \operatorname{Hom}_{\mathscr{O}_S\text{-alg}}(g_*\mathscr{O}_Y, f_*\mathscr{O}_X).$$

Proof. Since $g : Y \to S$ is an affine morphism, by 25 of the relative spectrum, there is a canonical isomorphism over S:

$$Y \cong \mathsf{Spec}_S(g_*\mathscr{O}_Y).$$

Let $\mathscr{B} = g_*\mathscr{O}_Y$. Substituting this into the geometric side, we apply the 6:

$$\operatorname{Hom}_S(X, Y) \cong \operatorname{Hom}_S(X, \mathsf{Spec}_S\mathscr{B}) \cong \operatorname{Hom}_{\mathscr{O}_S\text{-alg}}(\mathscr{B}, f_*\mathscr{O}_X).$$

This completes the proof. □

Corollary 30 Isomorphisms of Modules via Pushforward.

Let $f : X \to S$ be an **affine morphism**. Let $\mathscr{A} = f_*\mathscr{O}_X$.

i. For any quasi-coherent $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, we have a bijection:

$$\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \cong \operatorname{Hom}_{\mathscr{A}}(f_*\mathscr{F}, f_*\mathscr{G}).$$

(Here the RHS denotes morphisms of $\mathscr{A}$-modules on S).

ii. A morphism $\phi : \mathscr{F} \to \mathscr{G}$ is an isomorphism if and only if the induced morphism $f_*\phi : f_*\mathscr{F} \to f_*\mathscr{G}$ is an isomorphism.

Proof. Since f is affine, the 28 states that the functor $f_* : \mathsf{QCoh}(X) \to \mathsf{Mod}(\mathscr{A})$ is an equivalence of categories.

i. Any equivalence of categories is a **fully faithful** functor. This means precisely that the map on Hom-sets is a bijection.

ii. Any equivalence of categories is **conservative** (reflects isomorphisms). If $F : \mathcal{C} \to \mathcal{D}$ is an equivalence, then $F(\phi)$ is an iso in $\mathcal{D}$ if and only if ϕ is an iso in $\mathcal{C}$. Applying this to $F = f_*$ gives the result.

□

To fully appreciate the power of the relative spectrum Spec_S and the equivalence of categories, we examine three concrete examples ranging from geometry to arithmetic.

Example 31 Example 1: Geometric Vector Bundles.

The most important application of the relative spectrum is the construction of vector bundles.

Definition 32 Geometric Vector Bundle.

Let S be a scheme and let $\mathscr{E}$ be a locally free sheaf of rank n on S. The **geometric vector bundle** associated to $\mathscr{E}$ is defined as:

$$\mathbb{V}(\mathscr{E}) := \mathsf{Spec}_S\left(\mathrm{Sym}^\bullet_{\mathscr{O}_S}(\mathscr{E})\right)$$

where $\mathrm{Sym}^\bullet_{\mathscr{O}_S}(\mathscr{E}) = \mathscr{O}_S \oplus \mathscr{E} \oplus \mathrm{Sym}^2(\mathscr{E}) \oplus \dots$ is the symmetric algebra of $\mathscr{E}$.

Let's use our 6 to understand morphisms into $\mathbb{V}(\mathscr{E})$. Let $f : X \to S$ be any S-scheme. What is an S-morphism $g : X \to \mathbb{V}(\mathscr{E})$? According to the adjunction:

$$\mathrm{Hom}_S(X, \mathbb{V}(\mathscr{E})) \cong \mathrm{Hom}_{\mathscr{O}_S\text{-alg}}(\mathrm{Sym}^\bullet(\mathscr{E}), f_*\mathscr{O}_X).$$

By the universal property of the symmetric algebra, a morphism from $\mathrm{Sym}^\bullet(\mathscr{E})$ to an algebra $\mathscr{A}$ is uniquely determined by a linear map from $\mathscr{E}$ to $\mathscr{A}$. Thus:

$$\mathrm{Hom}_{\mathscr{O}_S\text{-alg}}(\mathrm{Sym}^\bullet(\mathscr{E}), f_*\mathscr{O}_X) \cong \mathrm{Hom}_{\mathscr{O}_S\text{-mod}}(\mathscr{E}, f_*\mathscr{O}_X).$$

Using the standard module adjunction (f^*, f_*), this is isomorphic to:

$$\mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{E}, \mathscr{O}_X).$$

This corresponds to the dual sheaf $(f^*\mathscr{E})^\vee$.

i. **Conclusion:** Giving a morphism $X \to \mathbb{V}(\mathscr{E})$ is equivalent to giving a section of the dual bundle on X.

ii. **Special Case (Sections):** If $X = S$ and $f = \mathrm{id}$, then sections $s : S \to \mathbb{V}(\mathscr{E})$ correspond to elements of $\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{E}, \mathscr{O}_S) = \Gamma(S, \mathscr{E}^\vee)$.

This example shows how the 6 turns a complex geometric question (maps between schemes) into a linear algebra calculation.

Example 33 Example 2: Normalization of a Singular Curve (Illustrating Relative Spectrum).

This example shows how we construct a new scheme solely by manipulating the sheaf of algebras.

Let k be a field and let $C = \operatorname{Spec} A$ be the cuspidal cubic curve, where $A = k[x, y]/(y^2 - x^3)$. Let $S = C$. We observe that the ring A is not integrally closed. Its integral closure in its field of fractions is the polynomial ring $k[t]$, via the map $x \mapsto t^2, y \mapsto t^3$. Let $B = k[t]$. We have a finite ring homomorphism $\phi : A \to B$. Let $\mathcal{A}$ be the sheaf associated to the A-module B. Since multiplication in B is compatible with A, $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras.

We define the **normalization** of C as:

$$\widetilde{C} := \mathsf{Spec}_S(\mathcal{A}).$$

Since S is affine, we know $\mathsf{Spec}_S(\mathcal{A}) \cong \operatorname{Spec}(\Gamma(S, \mathcal{A})) = \operatorname{Spec} B = \mathbb{A}^1_k$. The structure morphism $\pi : \widetilde{C} \to C$ corresponds to the map $t \mapsto (t^2, t^3)$, which is the parametrization of the cusp.

Here, $f_* \mathcal{O}_{\widetilde{C}}$ is exactly the sheaf $\mathcal{A}$. The morphism is affine (in fact, finite). This illustrates that "resolving a singularity" (in dimension 1) can be viewed as replacing the structure sheaf $\mathcal{O}_C$ with its integral closure $f_* \mathcal{O}_{\widetilde{C}}$ and taking Spec_S.

Example 34 Example 3: Modules over Number Rings (Illustrating Equivalence of Categories).

Let's look at an arithmetic example to understand 28.

Let $S = \operatorname{Spec} \mathbb{Z}$. Let $K = \mathbb{Q}(\sqrt{-1})$ be the Gaussian field. Its ring of integers is $\mathcal{O}_K = \mathbb{Z}[i]$. Let $X = \operatorname{Spec} \mathbb{Z}[i]$. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[i]$ induces an affine morphism $f : X \to S$.

Let $\mathcal{A} = f_* \mathcal{O}_X$. This is a coherent sheaf of algebras on $\operatorname{Spec} \mathbb{Z}$.

i. A quasi-coherent sheaf $\mathcal{F}$ on X corresponds to a $\mathbb{Z}[i]$-module M.

ii. The direct image $f_* \mathcal{F}$ corresponds to the $\mathbb{Z}$-module M (restriction of scalars).

The 29 says:

$$\mathsf{QCoh}(X) \cong \mathsf{Mod}(\mathcal{A}).$$

In algebraic terms, this means:

> "To give a module over $\mathbb{Z}[i]$ is equivalent to giving a module over $\mathbb{Z}$ equipped with a compatible action of the algebra $\mathbb{Z}[i]$."

While this sounds trivial algebraically, geometrically it implies that we can study the "arithmetic surface" X entirely by looking at objects on the "base line" S, provided we keep track of the extra algebra structure $\mathcal{A}$.

Corollary 35 Isomorphisms of Modules over Number Rings.

Let $\phi : M \to N$ be a homomorphism of $\mathbb{Z}[i]$-modules. As we stated: ϕ is an isomorphism if and only if it is an isomorphism of underlying abelian groups ($\mathbb{Z}$-modules).

6.5 The Functor of Points

Definition 36 The Functor of Points.

Let Sch denote the category of schemes and Set denote the category of sets. For any scheme $X \in \mathrm{Ob}(\mathsf{Sch})$, we define the *functor of points* of X, denoted by h_X, as the contravariant functor:

$$h_X : \mathsf{Sch}^{\mathrm{op}} \longrightarrow \mathsf{Set}$$

This functor acts on objects $Y \in \mathrm{Ob}(\mathsf{Sch})$ by sending them to the set of morphisms from Y to X:

$$h_X(Y) := \mathrm{Hom}_{\mathsf{Sch}}(Y, X).$$

We refer to elements of the set $h_X(Y)$ as the *Y-valued points* of X.

For a morphism $f : Y' \to Y$ in Sch, the functor applies the map $h_X(f) : h_X(Y) \to h_X(Y')$ defined by pre-composition:

$$h_X(f)(g) = g \circ f, \quad \text{for all } g \in \mathrm{Hom}_{\mathsf{Sch}}(Y, X).$$

Theorem 37 Yoneda Lemma and Embedding.

Let $\mathsf{Fun}(\mathsf{Sch}^{\mathrm{op}}, \mathsf{Set})$ be the category of presheaves on Sch (i.e., contravariant functors from schemes to sets), where morphisms are natural transformations. Consider the functor:

$$\Phi : \mathsf{Sch} \longrightarrow \mathsf{Fun}(\mathsf{Sch}^{\mathrm{op}}, \mathsf{Set}), \quad X \mapsto h_X.$$

The Yoneda Lemma asserts that for any scheme X and any functor $F \in \mathsf{Fun}(\mathsf{Sch}^{\mathrm{op}}, \mathsf{Set})$, there is a bijection:

$$\mathrm{Nat}(h_X, F) \cong F(X),$$

which is natural in both X and F. Specifically, taking $F = h_{X'}$ for another scheme X', we obtain:

$$\mathrm{Nat}(h_X, h_{X'}) \cong h_{X'}(X) = \mathrm{Hom}_{\mathsf{Sch}}(X, X').$$

Proof. The correspondence maps a natural transformation $\alpha : h_X \to h_{X'}$ to the element $\alpha_X(\mathrm{id}_X) \in h_{X'}(X)$. Conversely, a morphism $\phi : X \to X'$ determines a natural transformation via post-composition. The rigorous content of Yoneda's Lemma implies that the functor Φ is *fully faithful*.

Consequently, the category Sch is equivalent to the full subcategory of $\mathsf{Fun}(\mathsf{Sch}^{\mathrm{op}}, \mathsf{Set})$ consisting of *representable functors*. A functor F is representable if there exists a scheme X such that $F \cong h_X$. Under this equivalence, to give a morphism of schemes $X \to X'$ is exactly the same as giving a natural transformation $h_X \to h_{X'}$. □

Intuition 38 Geometric Implications.

This perspective represents a shift from viewing a scheme X as a locally ringed space $(|X|, \mathcal{O}_X)$ to viewing it as a "machine" that assigns sets to test objects. The scheme X is completely determined by the totality of its Y-valued points for all Y.

For example, if $X = \mathrm{Spec}(\mathbb{Z}[t_1, \ldots, t_n]/(f_1, \ldots, f_m))$, then for any ring A, the set of $(\mathrm{Spec}\, A)$-

valued points, $h_X(\operatorname{Spec} A)$, is naturally identified with the set of solutions to the system $f_i = 0$ in A^n. Thus, the functor of points formalizes the notion of a system of equations having solutions in arbitrary rings.

Proposition 39 Functorial Reconstruction of R-Schemes.

Let R be a commutative ring. Let Sch_R denote the category of schemes over R, and let Alg_R denote the category of commutative R-algebras.

Consider the functor $\Psi : \mathsf{Sch}_R \to \mathsf{Fun}(\mathsf{Alg}_R, \mathsf{Set})$ defined by assigning to an R-scheme X its covariant functor of points restricted to affine schemes:

$$h_X^{\text{aff}} : A \mapsto \operatorname{Hom}_{\mathsf{Sch}_R}(\operatorname{Spec} A, X).$$

Then, Ψ is fully faithful. Consequently, the category Sch_R is equivalent to the full subcategory of functors on Alg_R that are sheaves in the Zariski topology.

Proof. To prove that Ψ is fully faithful, we must show that for any two R-schemes X and Y, the map

$$\operatorname{Hom}_{\mathsf{Sch}_R}(X, Y) \to \operatorname{Nat}(h_X^{\text{aff}}, h_Y^{\text{aff}})$$

is bijective.

Injectivity (Faithfulness): Let $f, g : X \to Y$ be two morphisms such that they induce the same natural transformation on affine test objects. Let $\{U_i\}_{i \in I}$ be an affine open cover of X. For each i, the inclusion $\iota_i : U_i \hookrightarrow X$ corresponds to an element in $h_X^{\text{aff}}(\mathscr{O}_X(U_i))$. By hypothesis, the induced maps agree on these affine open sets, i.e., $f|_{U_i} = g|_{U_i}$. Since morphisms of schemes are determined locally, $f = g$.

Surjectivity (Fullness): Let $\alpha : h_X^{\text{aff}} \to h_Y^{\text{aff}}$ be a natural transformation. We need to construct a morphism $f : X \to Y$ inducing α. Let $\{U_i = \operatorname{Spec} A_i\}_{i \in I}$ be an affine open cover of X. The inclusion $U_i \hookrightarrow X$ is an element $u_i \in h_X^{\text{aff}}(A_i)$. The natural transformation α provides an element $\alpha_{A_i}(u_i) \in h_Y^{\text{aff}}(A_i)$, which corresponds to a morphism $f_i : U_i \to Y$.

We must verify that these local morphisms f_i glue together. Consider the intersection $U_i \cap U_j$. Since the intersection of affine opens is not necessarily affine (unless X is separated, but we do not assume this), we cover $U_i \cap U_j$ by open affines $\{V_{ijk} = \operatorname{Spec} B_{ijk}\}_k$. The naturality of α with respect to the restriction maps $A_i \to B_{ijk}$ ensures that $f_i|_{V_{ijk}} = f_j|_{V_{ijk}}$. By the sheaf property of morphisms, the f_i glue to a unique global morphism $f : X \to Y$.

Thus, X is completely determined by its functor of points restricted to affine R-algebras. □

Proposition 40 Determination by Noetherian Rings.

Let X be a locally Noetherian scheme over a Noetherian ring R. Let $\mathsf{NoethAlg}_R$ denote the full subcategory of Alg_R consisting of Noetherian R-algebras (i.e., finitely generated algebras).

Then X is determined by the restriction of its functor of points to $\mathsf{NoethAlg}_R$. That is, the restriction functor

$$\operatorname{Nat}(h_X^{\text{aff}}, h_Y^{\text{aff}}) \to \operatorname{Nat}(h_X|_{\mathsf{NoethAlg}_R}, h_Y|_{\mathsf{NoethAlg}_R})$$

is a bijection for locally Noetherian schemes X, Y.

Proof. The statement is equivalent to saying that the values of the functor $h_X^{\text{aff}}(A)$ for an arbitrary R-algebra A are determined by the values $h_X^{\text{aff}}(B)$ where B ranges over Noetherian algebras.

We rely on the property of algebraic limits. Any commutative R-algebra A can be written as the filtered colimit (direct limit) of its finitely generated (hence Noetherian, since R is Noetherian) subalgebras:

$$A = \varinjlim_{\lambda \in \Lambda} A_\lambda, \quad A_\lambda \in \mathrm{Ob}(\mathsf{NoethAlg}_R).$$

Since X is locally Noetherian, the structure morphism $X \longrightarrow \operatorname{Spec} R$ is locally of finite presentation. A key result in scheme theory (EGA IV_3, 8.14.2) states that for a scheme X locally of finite presentation over S, the functor of points preserves filtered colimits of affine rings. Explicitly:

$$h_X^{\text{aff}}(\varinjlim A_\lambda) = \mathrm{Hom}(\mathrm{Spec}(\varinjlim A_\lambda), X) \cong \varinjlim \mathrm{Hom}(\operatorname{Spec} A_\lambda, X) = \varinjlim h_X^{\text{aff}}(A_\lambda).$$

This isomorphism implies that the set of A-valued points is canonically determined by the system of A_λ-valued points. Therefore, any natural transformation defined on $\mathsf{NoethAlg}_R$ extends uniquely to all of Alg_R via this colimit extension. Since X is determined by h_X^{aff} on all algebras (by Proposition 39), it is determined by the restriction to Noetherian algebras. □

Definition 41 **Subfunctors and Fibered Products.**

Let Sch denote the category of schemes and Set the category of sets. Let $F : \mathsf{Sch}^{\mathrm{op}} \longrightarrow \mathsf{Set}$ be a functor.

A natural transformation $\alpha : G \longrightarrow F$ is called a *subfunctor* if for every scheme T, the map of sets $\alpha_T : G(T) \longrightarrow F(T)$ is injective. We often identify G with its image and write $G \subset F$.

Given functors F_1, F_2, F and morphisms $\alpha : F_1 \longrightarrow F$, $\beta : F_2 \longrightarrow F$, the *fibered product* $F_1 \times_F F_2$ is the functor defined by:

$$(F_1 \times_F F_2)(T) := \{(x, y) \in F_1(T) \times F_2(T) \mid \alpha_T(x) = \beta_T(y)\}.$$

This construction is the categorical limit in the functor category $\mathsf{Fun}(\mathsf{Sch}^{\mathrm{op}}, \mathsf{Set})$.

Definition 42 **Open and Closed Subfunctors.**

A subfunctor $G \subset F$ is called an *open subfunctor* if for every scheme Y and every morphism $\phi : h_Y \longrightarrow F$ (which, by Yoneda, corresponds to an element of $F(Y)$), the fibered product $G \times_F h_Y$ is representable by an open subscheme of Y.

Specifically, there exists an open subscheme $U \subset Y$ such that the projection $G \times_F h_Y \longrightarrow h_Y$ is isomorphic to the natural inclusion $h_U \hookrightarrow h_Y$.

Similarly, $G \subset F$ is a *closed subfunctor* if for every map $h_Y \longrightarrow F$, the pullback $G \times_F h_Y$ is representable by a closed subscheme of Y.

Proposition 43 Correspondence of Subschemes and Subfunctors.

Let X be a scheme and h_X its functor of points.

1. There is a one-to-one correspondence between open subschemes $U \subset X$ and open subfunctors $G \subset h_X$, given by $U \mapsto h_U$.
2. Similarly, closed subschemes $Z \subset X$ correspond bijectively to closed subfunctors of h_X.

Proof. We treat the open case; the closed case proceeds strictly analogously.

First, assume $U \subset X$ is an open subscheme. The inclusion $j : U \hookrightarrow X$ is a monomorphism, so $h_j : h_U \to h_X$ is a subfunctor. To verify it is an open subfunctor, let $\phi : h_Y \to h_X$ be any morphism, corresponding to a morphism of schemes $f : Y \to X$. The fibered product $h_U \times_{h_X} h_Y$ represents the functor of points of the fiber product of schemes $U \times_X Y$. Since open immersions are stable under base change, the projection $U \times_X Y \to Y$ is an open immersion identifying $U \times_X Y$ with the open subscheme $f^{-1}(U) \subset Y$. Thus, the pullback is representable by an open subscheme, satisfying the definition.

Conversely, let $G \subset h_X$ be an open subfunctor. Consider the identity morphism $\mathrm{id}_X \in h_X(X)$, which corresponds to the Yoneda embedding map $1 : h_X \to h_X$. By the definition of an open subfunctor, the pullback $G \times_{h_X} h_X$ must be representable by an open subscheme $U \subset X$. Since $G \times_{h_X} h_X$ is naturally isomorphic to G itself (as the intersection of a subset with the total set), we conclude that G is representable by the open subscheme U. The uniqueness of U follows from the Yoneda Lemma, which states that the representing object is unique up to unique isomorphism. □

Definition 44 Open Covering of a Functor.

A collection of open subfunctors $\{G_i \subset F\}_{i \in I}$ is called an *open covering* of the functor F if for every morphism $h_Y \to F$ from a representable functor, the resulting open subschemes $\{U_i \subset Y\}$ (where $h_{U_i} \cong G_i \times_F h_Y$) constitute a Zariski open cover of the scheme Y, i.e., $Y = \bigcup_{i \in I} U_i$.

Example 45 Sheaf Property vs. Set-Theoretic Union.

It is crucial to distinguish between an open covering of a functor and a union of sets of points. Let $X = \operatorname{Spec} \mathbb{Z}$ and consider the standard open cover given by two coprime integers, say $p = 2$ and $q = 3$. Since $(2, 3) = 1$, the principal open sets $U_p = D(2) = \operatorname{Spec} \mathbb{Z}[1/2]$ and $U_q = D(3) = \operatorname{Spec} \mathbb{Z}[1/3]$ form an open cover of X because $D(2) \cup D(3) = X$.

This induces an open covering of functors $\{h_{U_p} \to h_X, h_{U_q} \to h_X\}$. However, for a test scheme T, it is generally **false** that $h_X(T) = h_{U_p}(T) \cup h_{U_q}(T)$.

Consider the test object $T = \operatorname{Spec} \mathbb{Z} = X$. The set $h_X(T) = \operatorname{Hom}(X, X)$ contains the identity morphism id_X. Now examine the set $h_{U_p}(T) = \operatorname{Hom}(X, U_p)$. A morphism $f : X \to U_p$ must land inside the open set $U_p \subsetneq X$. The identity map id_X is surjective onto X, so its image contains the point (2), which is not in U_p. Thus, $\mathrm{id}_X \notin h_{U_p}(T)$. Similarly, $\mathrm{id}_X \notin h_{U_q}(T)$ because $(3) \notin U_q$.

Consequently,

$$\mathrm{id}_X \in h_X(T) \quad \text{but} \quad \mathrm{id}_X \notin h_{U_p}(T) \cup h_{U_q}(T).$$

This illustrates that the functor of points is a sheaf (where sections glue), not merely a disjoint union of its local values. The "covering" property is tested by pulling back to base schemes, not by checking set membership globally.

Definition 46 k-Rational Points.

Let X be a scheme over a field k, equipped with the structure morphism $\pi : X \longrightarrow \operatorname{Spec} k$. A *$k$-rational point* (or simply a k-point) of X is a morphism of schemes $x : \operatorname{Spec} k \longrightarrow X$ such that $\pi \circ x = \mathrm{id}_{\operatorname{Spec} k}$.

In the language of the functor of points, the set of k-rational points is denoted by $X(k)$. It is precisely the set $h_X(\operatorname{Spec} k) = \mathrm{Hom}_{\mathsf{Sch}_k}(\operatorname{Spec} k, X)$. If $X = \operatorname{Spec} A$ is an affine scheme over k, where A is a k-algebra, then a k-rational point corresponds to a k-algebra homomorphism $\phi : A \longrightarrow k$. Geometrically, this corresponds to a closed point $p \in X$ whose residue field $\kappa(p)$ is isomorphic to k via the structure map.

Example 47 Rational Points and Galois Action.

Let X be a scheme defined over a field k, and let K/k be a Galois extension with Galois group $\Gamma = \mathrm{Gal}(K/k)$. The set of K-rational points, $X(K) = \mathrm{Hom}_{\mathsf{Sch}_k}(\operatorname{Spec} K, X)$, carries a natural action of the group Γ.

We define this action as follows: For any automorphism $\sigma \in \Gamma$, there is an induced isomorphism of schemes $\operatorname{Spec}(\sigma) : \operatorname{Spec} K \longrightarrow \operatorname{Spec} K$. Given a K-point $x \in X(K)$, which is a morphism $x : \operatorname{Spec} K \longrightarrow X$ satisfying the structural constraint, the group element σ acts on x to produce a new point x^σ. This new point is defined by the pre-composition $x^\sigma = x \circ \operatorname{Spec}(\sigma)$. Since the structure morphism $X \longrightarrow \operatorname{Spec} k$ is invariant under the base change of the spectrum of the field, this composition remains a valid point over $\operatorname{Spec} k$, though structurally it is viewed as a twisted map relative to the K-structure.

Consider the specific case of the special linear group $X = \mathrm{SL}_{2,\mathbb{F}_q}$ over a finite field $k = \mathbb{F}_q$. The set of $\mathbb{F}_q$-rational points, $X(\mathbb{F}_q)$, is the finite group of 2×2 matrices with entries in $\mathbb{F}_q$ and determinant 1. To count these points, we consider the action of SL_2 on the non-zero vectors of $\mathbb{F}_q^2$. The order is calculated by observing that the first column of such a matrix can be any non-zero vector ($q^2 - 1$ choices), and the second column must satisfy the determinant condition, leaving q choices. Thus, the cardinality is $|SL_2(\mathbb{F}_q)| = q(q^2 - 1)$. If we consider the base extension to the algebraic closure $\bar{\mathbb{F}}_q$, the Frobenius automorphism $\mathrm{Frob}_q : x \mapsto x^q$ generates a topological cyclic group acting on $X(\bar{\mathbb{F}}_q)$, and the fixed points of this action are precisely the original rational points $X(\mathbb{F}_q)$.

Definition 48 Group Schemes.

A *group scheme* over a base scheme S is a scheme G equipped with morphisms of S-schemes:

$$m : G \times_S G \longrightarrow G \quad \text{(multiplication)},$$

$$i : G \longrightarrow G \quad \text{(inverse)},$$

$$e : S \to G \quad \text{(identity section)},$$

such that these morphisms satisfy the standard categorical axioms of a group object. Specifically, the multiplication must be associative, e must act as a two-sided identity for m, and i must provide a two-sided inverse.

Alternatively, using the functor of points, a group scheme is a scheme G such that for every S-scheme T, the set of points $G(T)$ carries a group structure that is functorial in T. That is, the functor $h_G : (\mathsf{Sch}_S)^{\mathrm{op}} \to \mathsf{Set}$ factors through the category of groups, denoted Grp. The Yoneda Lemma guarantees that these two definitions—providing the morphisms (m, i, e) or providing the functorial group structure—are equivalent.

Example 49 Algebraic Groups and Lie Theory.

A prominent class of group schemes are *linear algebraic groups*, such as the general linear group $\mathrm{GL}_{n,k} = \operatorname{Spec} k[x_{11}, \ldots, x_{nn}, \det^{-1}]$ and its closed subgroups (like the orthogonal group O_n or symplectic group Sp_{2n}).

The connection between group schemes and Lie theory is realized through the tangent space at the identity. Let G be a group scheme over a field k, and let $e \in G(k)$ be the identity point. The *Lie algebra* of G, denoted $\mathrm{Lie}(G)$ or $\mathfrak{g}$, is defined as the Zariski tangent space at the identity, $T_e(G)$. This vector space can be identified with the kernel of the map $G(k[\epsilon]/(\epsilon^2)) \to G(k)$ induced by $\epsilon \mapsto 0$.

The group law $m : G \times G \to G$ induces a Lie bracket structure on $\mathfrak{g}$ via the differentiation of the adjoint representation. This establishes a functor from the category of group schemes (or algebraic groups) to the category of Lie algebras, generalizing the classical correspondence between Lie groups and Lie algebras in differential geometry.

CHAPTER 7

Global Properties of Morphisms of Schemes

7.1 Basic Definitions

Definition 1 Finiteness and Global Properties of Morphisms.

Let $f : X \to Y$ be a morphism of schemes.

(i) **Locally of Finite Type:** The morphism f is *locally of finite type* if there exists a covering of Y by open affine subsets $V_j = \mathrm{Spec}(B_j)$ such that for each j, $f^{-1}(V_j)$ can be covered by open affine subsets $U_{ij} = \mathrm{Spec}(A_{ij})$, where each A_{ij} is a **finitely generated** B_j-algebra.

(ii) **Of Finite Type:** The morphism f is *of finite type* if it is locally of finite type and **quasi-compact**. (Equivalently, $f^{-1}(V_j)$ can be covered by a **finite** number of such U_{ij}).

(iii) **Affine Morphism:** The morphism f is *affine* if the inverse image of every open affine subset of Y is an **open affine subset** of X.

(iv) **Integral Morphism:** The morphism f is *integral* if it is affine and, for every open affine $V = \mathrm{Spec}(B) \subseteq Y$, the ring A in $f^{-1}(V) = \mathrm{Spec}(A)$ is an **integral** B**-extension**. That is, every element $a \in A$ satisfies a monic polynomial equation $a^n + b_{n-1}a^{n-1} + \cdots + b_0 = 0$ with coefficients $b_i \in B$.

(v) **Finite Morphism:** The morphism f is *finite* if it is affine and, for every open affine $V = \mathrm{Spec}(B) \subseteq Y$, the ring A in $f^{-1}(V) = \mathrm{Spec}(A)$ is **finitely generated as a** B**-module**.

(vi) **Proper Morphism:** The morphism f is *proper* if it is of finite type, separated, and **universally closed** (i.e., for any morphism $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is a closed map of topological spaces).

(vii) **Projective Morphism:** The morphism f is *projective* if it factors as a **closed immersion** followed by a projection from a projective space over Y:

$$X \hookrightarrow \mathbb{P}^n_Y \to Y$$

for some $n \geq 0$.

In the theory of schemes, we often study properties of morphisms $\mathcal{P}$ (e.g., being finite, flat, smooth, proper). To formalize this, we define standard meta-properties that a property $\mathcal{P}$ might possess.

Definition 2 Properties of Morphisms.

Let $\mathcal{P}$ be a property of morphisms of schemes.

(i) **Local on the target (Target-local):** A morphism $f : X \to Y$ has property $\mathcal{P}$ if and only if for any open covering $\{V_i\}$ of Y, the induced morphism $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ has property $\mathcal{P}$ for each i.

(ii) **Stable under composition:** If $f : X \to Y$ and $g : Y \to Z$ have $\mathcal{P}$, then $g \circ f$ has $\mathcal{P}$.

(iii) **Stable under base change:** If $f : X \to Y$ has $\mathcal{P}$, then for any morphism $Y' \to Y$, the projection $f' : X \times_Y Y' \to Y'$ has $\mathcal{P}$.

(iv) **Stable under products:** If $f : X \to Y$ and $f' : X' \to Y'$ have $\mathcal{P}$, then $f \times f' : X \times X' \to Y \times Y'$ has $\mathcal{P}$ (over the appropriate base product).

(v) **Cancellation (or Cancellation of the right):** If $g \circ f$ has $\mathcal{P}$ and g is separated (or satisfies some other mild condition), then f has $\mathcal{P}$. (Note: Specific definitions vary, but this is a common form).

To check if a property is local on the target, we often need to refine covers. The following lemmas guarantee that the intersection of affine charts from different covers is well-behaved.

Lemma 3 Common Principal Open Neighborhood.

Let X be a scheme. Let $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ be two affine open subsets of X. For any point $x \in U \cap V$, there exists an open neighborhood W of x such that $W \subseteq U \cap V$, and W is a principal open set (distinguished open) in **both** U and V.

That is, there exist $f \in A$ and $g \in B$ such that:

$$W = D_U(f) = D_V(g).$$

Proof. Since U and V are open in X, their intersection $U \cap V$ is open in X.

Step 1: Shrinking in U. Consider x as a point in the affine scheme U. Since $U \cap V$ is an open neighborhood of x inside U, and the principal open sets $\{D_U(h)\}_{h \in A}$ form a basis for the Zariski topology on U, there exists an element $h \in A$ such that:

$$x \in D_U(h) \subseteq U \cap V.$$

Step 2: Shrinking in V. Now consider the set $D_U(h)$. It is an open subset of X contained in V. Since $\{D_V(k)\}_{k \in B}$ forms a basis for the topology on V, there exists an element $k \in B$ such that:

$$x \in D_V(k) \subseteq D_U(h).$$

Step 3: Finding the common set. We now have $x \in D_V(k) \subseteq D_U(h)$. Note that $D_V(k)$ is a principal open set in V, hence it is affine (isomorphic to $\operatorname{Spec} B_k$). Since $D_V(k)$ is a subset of U (specifically a subset of $D_U(h)$), we can view the element $h \in A$ as a function on $D_V(k)$ by restricting the section. More precisely, let $\rho : A \to \Gamma(D_V(k), \mathcal{O}_X) = B_k$ be the restriction homomorphism. Let $\bar{h} = \rho(h) \in B_k$.

The condition $D_V(k) \subseteq D_U(h)$ implies that for any point $y \in D_V(k)$, the function h does not vanish at y (i.e., $h_y \notin \mathfrak{m}_y$). However, we want to perform a further refinement to ensure the set is principal in U. Let's reverse the inclusion logic for a cleaner construction: The set $D_V(k)$ is open in U. Since $D_U(h)$ is affine (isomorphic to Spec A_h), and $D_V(k)$ is open in $D_U(h)$, $D_V(k)$ can be covered by principal opens of A_h. Thus, there exists $\ell \in A_h$ such that:

$$x \in D_{A_h}(\ell) \subseteq D_V(k).$$

Recall that a principal open of a principal open is principal. Specifically, ℓ can be written as a/h^n. The set $D_{A_h}(\ell)$ corresponds to the locus in $D_U(h)$ where $a/h^n \neq 0$, which is the locus where $a \neq 0$ and $h \neq 0$. This is exactly $D_U(a \cdot h)$. Let $f = a \cdot h \in A$. So we have:

$$W = D_U(f) \subseteq D_V(k).$$

Step 4: Making it principal in V. Now $W = D_U(f)$ is a subset of $D_V(k)$. Is W principal in V? Consider f restricted to $D_V(k) = \operatorname{Spec} B_k$. Let this restriction be $f' \in B_k$. The set W is the locus inside $D_V(k)$ where f does not vanish. Algebraically, this is $D_{B_k}(f')$. Again, by transitivity of principal opens, $D_{B_k}(f')$ is isomorphic to a principal open set $D_V(g)$ for some $g \in B$ (specifically, if $f' = b/k^m$, then $D_{B_k}(f') = D_V(b \cdot k)$).

Thus, $W = D_U(f) = D_V(g)$ is a principal open set in both U and V. □

Lemma 4 Preimage of Principal Open is Principal Open.

Let $\phi : A \to B$ be a ring homomorphism, inducing a morphism of affine schemes $f : \operatorname{Spec} B \to \operatorname{Spec} A$. Let $h \in A$. Then the inverse image of the principal open set $D(h) \subseteq \operatorname{Spec} A$ is a principal open set in $\operatorname{Spec} B$. Specifically:

$$f^{-1}(D(h)) = D(\phi(h)).$$

Proof. This follows directly from the definition of the continuous map induced by a ring homomorphism. Recall that for a point $\mathfrak{q} \in \operatorname{Spec} B$, its image is $f(\mathfrak{q}) = \phi^{-1}(\mathfrak{q}) \in \operatorname{Spec} A$. We have:

$$\begin{aligned} \mathfrak{q} \in f^{-1}(D(h)) &\iff f(\mathfrak{q}) \in D(h) \\ &\iff h \notin f(\mathfrak{q}) \\ &\iff h \notin \phi^{-1}(\mathfrak{q}) \\ &\iff \phi(h) \notin \mathfrak{q} \\ &\iff \mathfrak{q} \in D(\phi(h)). \end{aligned}$$

Thus, the preimage is exactly the principal open set defined by the image of the element h. □

We introduce the philosophy of "relative geometry": properties of schemes are generalized to properties of morphisms.

Proposition 5 Target-Locality of Quasi-compactness and Affineness.

Let $f : X \to Y$ be a morphism of schemes.

(1) **Quasi-compactness is local on the target:** f is quasi-compact if and only if there exists an open covering $\{V_i\}$ of Y such that each restriction $f^{-1}(V_i) \to V_i$ is quasi-compact.

(2) **Affineness is local on the target:** f is an affine morphism if and only if there exists an open covering $\{V_i\}$ of Y by affine open sets such that each $f^{-1}(V_i)$ is affine.

Proof. **(1) Proof for Quasi-compactness:** The "only if" direction is immediate from the definition (since open subsets of quasi-compact sets are not necessarily quasi-compact, but the definition requires checking preimages of *quasi-compact* opens, and subsets of the cover can be handled).

For the "if" direction: Assume we have a cover $\{V_i\}$ such that $f^{-1}(V_i) \to V_i$ is quasi-compact. Let $V \subseteq Y$ be any quasi-compact open subset. We must show $f^{-1}(V)$ is quasi-compact. Since V is quasi-compact, we can cover it by a **finite** number of affine open sets $\{W_j\}_{j=1}^{m}$. It suffices to show that $f^{-1}(W_j)$ is quasi-compact for each j, as a finite union of quasi-compact sets is quasi-compact. Thus, we reduce to the case where the target V is affine.

Since $\{V_i\}$ covers Y, the sets $\{V \cap V_i\}$ cover V. Since V is affine (hence quasi-compact) and the basis of the topology consists of standard open sets (which are affine), we can cover V by finitely many standard open sets $D(g_k)$ such that each $D(g_k)$ is contained in some V_{i_k}. The morphism restricted to V_{i_k} is quasi-compact, and $D(g_k)$ is a quasi-compact open subset of V_{i_k}. Therefore, its preimage $f^{-1}(D(g_k))$ is quasi-compact. Now, $f^{-1}(V) = \bigcup_k f^{-1}(D(g_k))$. Since this is a finite union of quasi-compact spaces, $f^{-1}(V)$ is quasi-compact.

(2) Proof for Affineness: The "only if" part is trivial. For the "if" direction: Assume $\{V_i\}$ is an affine open cover of Y such that $f^{-1}(V_i)$ is affine. Let $V \subseteq Y$ be *any* affine open set. We need to show that $f^{-1}(V)$ is an affine scheme.

Since V is affine, we can cover it by finitely many standard principal open sets $D_V(h_k)$ such that each $D_V(h_k)$ is contained in some V_{i_k}. Crucially, by our **Lemma 3** (Transition of Affine Covers), we can ensure that these sets are principal open sets in V_{i_k} as well. Let $U_k = f^{-1}(D_V(h_k))$. Since $D_V(h_k) \subseteq V_{i_k}$ is a principal open set of the affine scheme V_{i_k}, and $f^{-1}(V_{i_k})$ is affine (by hypothesis), the preimage U_k is a principal open set of an affine scheme (by **Lemma 4**). Therefore, each U_k is an affine scheme.

Now we look at the morphism $f|_{f^{-1}(V)} : f^{-1}(V) \to V$. We have a finite cover of V by basic opens $D_V(h_k)$ (where h_k generate the unit ideal in $\Gamma(V, \mathscr{O}_Y)$). The preimages U_k cover $X' = f^{-1}(V)$. Consider the pullback of the functions h_k to X', denoted by $h'_k = f^\sharp(h_k)$. We have $U_k = (X')_{h'_k}$. Since the h_k generate the unit ideal in the base ring, their pullbacks h'_k generate the unit ideal in $\Gamma(X', \mathscr{O}_{X'})$.

We are now in the situation: X' is covered by finitely many open sets $(X')_{h'_k}$, each of which is affine (U_k), and the h'_k generate the unit ideal. By the **Affine Criterion (Proposition 21)**, X' is an affine scheme. □

Lemma 6 Affine Communication Lemma (Algebraic).

Let $\phi : A \to B$ be a ring homomorphism. Let $f_1, \dots, f_n \in A$ be elements generating the unit ideal, i.e., $(f_1, \dots, f_n) = A$. Let $\mathcal{P}$ be a property of rings or ring homomorphisms. If the induced homomorphisms $\phi_i : A_{f_i} \to B_{f_i}$ satisfy $\mathcal{P}$ for all $i = 1, \dots, n$, then $\phi : A \to B$ satisfies $\mathcal{P}$.

We prove this for the following properties $\mathcal{P}$:

(a) **Finite Module:** B is a finitely generated A-module.

(b) **Finite Algebra:** B is a finitely generated A-algebra (finite type).

(c) **Quotient:** ϕ is surjective (isomorphic to a quotient).

(d) **Isomorphism:** ϕ is an isomorphism (isomorphic to the localization by 1).

Proof. The algebraic engine behind all these proofs is the **Partition of Unity**. Since $\sum Af_i = A$, there exist $a_i \in A$ such that $\sum_{i=1}^{n} a_i f_i = 1$.

(a) Finite Module. Suppose B_{f_i} is a finitely generated A_{f_i}-module. We can choose a finite set of generators $x_{ij} \in B_{f_i}$. By clearing denominators (multiplying by high powers of f_i), we can assume the generators come from B. Let $S \subset B$ be the finite set collecting all these numerators for all i. Let $M \subseteq B$ be the A-submodule generated by S. We claim $M = B$. It suffices to show the inclusion is surjective locally. The localization M_{f_i} is the A_{f_i}-submodule of B_{f_i} generated by S. By construction, S contains a generating set for B_{f_i}, so $M_{f_i} = B_{f_i}$. Since being zero is a local property (for the module B/M), $(B/M)_{f_i} = 0$ for all i implies $B/M = 0$, so $M = B$.

(b) Finite Algebra. The logic is identical to (a). Suppose B_{f_i} is generated as an A_{f_i}-algebra by finitely many elements. We collect the numerators of these generators into a finite set $S \subset B$. Consider the A-algebra $C = A[S] \subseteq B$ (image of the polynomial ring). Locally, $C_{f_i} \to B_{f_i}$ is surjective. Thus the cokernel module is locally zero, hence globally zero. So $A[S] \to B$ is surjective.

(c) Quotient (Surjectivity). Suppose $A_{f_i} \to B_{f_i}$ is surjective for all i. Let $C = \text{Coker}(\phi) = B/\text{Im}(A)$. Localization is exact, so $(\text{Coker}(\phi))_{f_i} \cong \text{Coker}(\phi_i)$. By hypothesis, ϕ_i is surjective, so $(\text{Coker}(\phi))_{f_i} = 0$. Since C is an A-module and f_i generate the unit ideal, $C_{f_i} = 0 \implies C = 0$. Thus ϕ is surjective.

(d) Isomorphism. We treat "isomorphic to localization of A" in the strongest sense: verifying that ϕ is an isomorphism (corresponding to the trivial localization). We need to show ϕ is injective and surjective.

- *Surjectivity:* Follows from (c).
- *Injectivity:* Let $K = \text{Ker}(\phi)$. Exactness of localization implies $K_{f_i} \cong \text{Ker}(\phi_i)$. Since $\phi_i : A_{f_i} \to B_{f_i}$ is an isomorphism, $K_{f_i} = 0$. Let $x \in K$. Then for each i, $x/1 = 0$ in K_{f_i}, which means there exists an integer N_i such that $f_i^{N_i} x = 0$ in A. Let $N = \max(N_i)$. Then $f_i^N x = 0$ for all i. Since $(f_1, \dots, f_n) = A$, the powers $(f_1^N, \dots, f_n^N)$ also generate the unit ideal (radicals!). Thus there exist b_i such that $\sum b_i f_i^N = 1$. Multiplying by x: $x = (\sum b_i f_i^N)x = \sum b_i (f_i^N x) = 0$. Thus $K = 0$, and ϕ is injective.

□

The general strategy to prove that a property $\mathcal{P}$ is local on the target is as follows:

1. **Restriction (⇒):** Usually follows from the definition or stability under base change.
2. **Gluing (⇐):** Assume Y has an open cover $\{V_i\}$ such that $f|_{f^{-1}(V_i)}$ has $\mathcal{P}$. We must show f has $\mathcal{P}$. We typically reduce to the case where Y is affine, cover it by principal opens contained in V_i, apply the algebraic lemma, and conclude for Y.

Proposition 7 Target-Locality of Open Immersions.

A morphism $f : X \to Y$ is an **open immersion** if and only if there exists an open covering $\{V_i\}$ of Y such that each induced morphism $f_i : f^{-1}(V_i) \to V_i$ is an open immersion.

Proof. The "only if" part is trivial. We prove the "if" part. Suppose each f_i is an open immersion. By definition, f_i induces an isomorphism from $f^{-1}(V_i)$ to an open subscheme $U_i \subseteq V_i$.

1. Topological Aspect: The image set $f(X) = \bigcup_i f(f^{-1}(V_i)) = \bigcup_i U_i$. Since each U_i is open in V_i and V_i is open in Y, U_i is open in Y. Thus $f(X)$ is open in Y. Since f is a homeomorphism locally on the target (restricted to open sets U_i), and being a homeomorphism is a local property, f is a homeomorphism onto its image.

2. Sheaf Aspect: We need to show the comorphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism on the open set $f(X)$. This is a local question. For any point $y \in f(X)$, $y \in V_i$ for some i. Restricted to V_i, $f^\sharp|_{V_i}$ corresponds to the map $\mathcal{O}_{V_i} \to (f_i)_*\mathcal{O}_{f^{-1}(V_i)}$. Since f_i is an open immersion, this is an isomorphism over the image U_i. Since $f^\sharp$ is locally an isomorphism onto the image, it is globally an isomorphism onto the image. Thus f is an open immersion. □

Proposition 8 Target-Locality of Closed Immersions.

A morphism $f : X \to Y$ is a **closed immersion** if and only if there exists an open covering $\{V_i\}$ of Y such that each $f_i : f^{-1}(V_i) \to V_i$ is a closed immersion.

Proof. **1. Topological Aspect:** The image $f(X)$ is closed in Y if and only if $f(X) \cap V_i$ is closed in V_i for all i. Since f_i is a closed immersion, its image is closed in V_i. Thus $f(X)$ is closed. Similarly, f is a homeomorphism onto its image.

2. Algebraic Aspect (Surjectivity): We need to show $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. This is a morphism of sheaves, so surjectivity is local. On each V_i, the map is $(f_i)^\sharp : \mathcal{O}_{V_i} \to (f_i)_*\mathcal{O}_{f^{-1}(V_i)}$. Since f_i is a closed immersion, this map is surjective. Being locally surjective implies global surjectivity. Thus f is a closed immersion.

Note: We implicitly used the fact that if f is a closed immersion locally, it is affine locally (by the Affine Communication Lemma part (c) - surjectivity), so X is affine over affine opens of Y. □

Proposition 9 Target-Locality of Immersions.

A morphism $f : X \to Y$ is an **immersion** if and only if it is so locally on the target.

Proof. Definition: f is an immersion if it factors as $X \xrightarrow{j} U \xrightarrow{i} Y$, where j is a closed immersion and i is an open immersion. This is equivalent to saying f induces an isomorphism from X onto a *locally closed* subscheme of Y. A subset $Z \subseteq Y$ is locally closed if and only if $Z \cap V_i$ is locally closed in V_i for an open cover $\{V_i\}$. The isomorphism condition is checked locally as in the previous propositions. □

Proposition 10 Target-Locality of Locally Finite Type.

A morphism $f : X \to Y$ is **locally of finite type** if and only if it is so locally on the target.

Proof. ($\Leftarrow$) Assume $\{V_i\}$ is an open cover of Y and f_i is locally of finite type. Let $V = \operatorname{Spec} A$ be *any* affine open subset of Y. We need to show that $f^{-1}(V)$ can be covered by affine open sets $\operatorname{Spec} B_k$ where each B_k is a finitely generated A-algebra.

First, cover V by finitely many principal open sets $D(h_j)$ such that each $D(h_j) \subseteq V_{i_j}$ for some index i_j. The restriction $f : f^{-1}(D(h_j)) \to D(h_j)$ is locally of finite type (since it restricts from f_{i_j}). Since $D(h_j)$ is affine (isomorphic to $\operatorname{Spec} A_{h_j}$), $f^{-1}(D(h_j))$ can be covered by affine opens $U_{j,k} = \operatorname{Spec} C_{j,k}$ where $C_{j,k}$ is a finitely generated A_{h_j}-algebra.

Since A_{h_j} is a finitely generated A-algebra, and $C_{j,k}$ is a f.g. A_{h_j}-algebra, by transitivity, $C_{j,k}$ is a finitely generated A-algebra. The collection $\{U_{j,k}\}$ covers $f^{-1}(V)$, and on each piece, the ring is a f.g. A-algebra. Thus f is locally of finite type. □

Proposition 11 Target-Locality of Finite Type.

A morphism $f : X \to Y$ is **of finite type** if and only if it is so locally on the target.

Proof. By definition, f is of finite type if it is **locally of finite type** and **quasi-compact**. We have proved that "locally of finite type" is target-local (Proposition 10). We have proved that "quasi-compactness" is target-local (previous section). The conjunction of two target-local properties is target-local. □

Proposition 12 Target-Locality of Integral Morphisms.

A morphism $f : X \to Y$ is **integral** if and only if it is so locally on the target. (Recall: f is integral if f is affine and for any affine open $V = \operatorname{Spec} A \subseteq Y$, if $f^{-1}(V) = \operatorname{Spec} B$, then B is integral over A via $f^\sharp$).

Proof. We have already established that **affineness** is local on the target (previous section). Thus, we may assume $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ are affine, and we have a cover $Y = \bigcup D(g_i)$ such that the induced maps $A_{g_i} \to B_{g_i}$ are integral ring homomorphisms (where g_i generate the unit ideal). We must show $A \to B$ is integral.

Algebraic Lemma (Integrality Descent): Let $b \in B$. We want to find a monic polynomial in $A[x]$ satisfied by b. For each i, the image of b in B_{g_i}, denoted $b/1$, is integral over A_{g_i}.

Thus, there exists an equation in $(A_{g_i})[x]$:

$$(b/1)^n + \frac{a_{n-1}}{g_i^k}(b/1)^{n-1} + \cdots + \frac{a_0}{g_i^k} = 0 \quad \text{in } B_{g_i}.$$

Multiplying by a sufficiently high power of g_i (say g_i^N where N is large enough to clear denominators and handle the localization map kernel), we get an equation in B:

$$g_i^N b^n + (\text{terms with lower powers of } b) = 0.$$

This is not yet a monic equation for b. However, this implies that for every i, b is integral over A_{g_i}... actually, there is a cleaner standard trick.

Consider the A-subalgebra $A[b] \subseteq B$. It is a finite A-module if and only if b is integral. Let $M = A[b]$. We want to show M is a finite A-module. We know that $M_{g_i} = A_{g_i}[b] \subseteq B_{g_i}$. Since b is integral over A_{g_i}, $A_{g_i}[b]$ is a finitely generated A_{g_i}-module. By the **Algebraic Lemma (Finite Module)** proven in the previous section, if M is an A-module such that M_{g_i} is finitely generated over A_{g_i} for a cover $\{g_i\}$, then M is finitely generated over A. Therefore, $A[b]$ is a finite A-module, which implies b is integral over A. Since b was arbitrary, B is integral over A. □

Proposition 13 Target-Locality of Finite Morphisms.

A morphism $f : X \to Y$ is **finite** if and only if it is so locally on the target.

Proof. By definition, f is finite if it is **affine** and for any affine open $V = \operatorname{Spec} A \subseteq Y$ with $f^{-1}(V) = \operatorname{Spec} B$, B is a **finite** (finitely generated) A-module.

1. We established Affineness is target-local.

2. We established that the property "$\phi : A \to B$ makes B a finite A-module" satisfies the algebraic descent condition (Algebraic Lemma part (a)).

Combining these: Assume local finiteness. Let $V = \operatorname{Spec} A \subseteq Y$. Since f is affine (by step 1), $f^{-1}(V) = \operatorname{Spec} B$. We cover V by $D(g_i)$ derived from the cover where f is known to be finite. Thus B_{g_i} is a finite A_{g_i}-module for all i. By the Algebraic Lemma, B is a finite A-module. Thus f is finite. □

We examine the stability of the following properties under composition, base change, and products:

- **Open Immersion**, **Closed Immersion**, **Immersion**
- **Locally of Finite Type**, **Finite Type**
- **Integral**, **Finite**

Proposition 14 Stability under Composition.

Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If f and g both have property $\mathcal{P}$, then the composition $g \circ f : X \to Z$ has property $\mathcal{P}$, where $\mathcal{P}$ is any of the properties listed above.

Proof. **1. Open/Closed Immersions:**

- **Open Immersion:** If f and g are open immersions, they are homeomorphisms onto open subsets. The composition of homeomorphisms onto open sets is a homeomorphism onto an open set (since an open subset of an open subset is open). The sheaf map is clearly an isomorphism.
- **Closed Immersion:** The composition of homeomorphisms onto closed sets is a homeomorphism onto a closed set (closed in subspace topology of a closed set is closed). Algebraically, if $C \to B$ and $B \to A$ are surjective ring homomorphisms, their composition $C \to A$ is surjective. Since closed immersions correspond locally to surjective ring maps, the property holds.

2. Immersion: An immersion is a composition of an open immersion followed by a closed immersion (or vice-versa, or an isomorphism onto a locally closed subscheme). Topologically, a locally closed subset of a locally closed subset is locally closed (intersection of open and closed). Thus $g \circ f$ maps X homeomorphically onto a locally closed subset of Z, and the sheaf map is surjective on stalks (locally generated by restriction).

3. Locally of Finite Type and Finite Type:

- **Locally of Finite Type:** Reduce to affine case. Let $Z = \operatorname{Spec} C$, $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$. g corresponds to $\psi : C \to B$ (finite type), f corresponds to $\phi : B \to A$ (finite type). If B is a finitely generated C-algebra (generators b_i) and A is a finitely generated B-algebra (generators a_j), then A is generated as a C-algebra by $\{\phi(b_i)\} \cup \{a_j\}$. Thus A is a finitely generated C-algebra.
- **Finite Type:** Finite Type = Locally Finite Type + Quasi-compact. The composition of quasi-compact morphisms is quasi-compact (preimage of quasi-compact is quasi-compact). Thus stability holds.

4. Integral and Finite:

- **Integral:** Let $C \to B \to A$. If B is integral over C and A is integral over B, then A is integral over C (Transitivity of Integral Dependence).
- **Finite:** Finite = Integral + Finite Type. Alternatively, in the affine case, if B is a finite C-module and A is a finite B-module, then A is a finite C-module.

□

Proposition 15 Stability under Base Change.

Let $f : X \to Y$ have property $\mathcal{P}$. Let $Y' \to Y$ be any morphism. Then the base change $f' : X \times_Y Y' \to Y'$ has property $\mathcal{P}$.

Proof. **Reduction Strategy:** Since all listed properties are **local on the target**, we can reduce the problem to the affine case.

1. Cover Y' by affine open sets $\{V_i'\}$. It suffices to show f' has $\mathcal{P}$ over each V_i'.
2. Each V_i' maps to Y. We can cover the image (or rather, refine the cover of Y' such that each piece maps into an affine of Y) by affine open sets $\{V_j\}$ of Y.
3. Thus, we reduce to the case where $Y = \operatorname{Spec} A$ and $Y' = \operatorname{Spec} A'$ are affine.
4. Furthermore, since the properties are stable under restricting to open subsets of X (local on source), or we can cover X by affines, we may assume $X = \operatorname{Spec} B$.
5. The base change $X' = X \times_Y Y'$ is affine, given by $\operatorname{Spec}(B \otimes_A A')$.

We now verify the algebraic properties for the ring map $\phi' : A' \to B \otimes_A A'$ induced by $\phi : A \to B$.

1. Open/Closed Immersions:

- **Closed Immersion:** $\phi : A \to B$ is surjective. Since the tensor product is right exact, $A' \to B \otimes_A A'$ is surjective (specifically, $B \cong A/I \implies B \otimes_A A' \cong A'/IA'$). Thus f' is a closed immersion.
- **Open Immersion:** This is stable by the topological construction of the fiber product. The preimage of an open set under a continuous map (the projection) is open.

2. Finite Type / Locally Finite Type: If B is a finitely generated A-algebra (generators $x_1, \dots, x_n$), then $B \otimes_A A'$ is generated as an A'-algebra by the elements $x_1 \otimes 1, \dots, x_n \otimes 1$. Thus the property is preserved. (Quasi-compactness is also stable under base change).

3. Integral: If B is integral over A, then every element $b \in B$ satisfies a monic polynomial with coefficients in A. Consider elements of the form $b \otimes 1$ in $B \otimes_A A'$. They satisfy the same polynomials (via the map $A \to A'$). Since $B \otimes_A A'$ is generated as an A'-algebra by elements of the form $b \otimes 1$, and the set of integral elements forms a subring, the whole ring $B \otimes_A A'$ is integral over A'.

4. Finite: If B is a finitely generated A-module (generators $m_1, \dots, m_k$), then $B \otimes_A A'$ is generated as an A'-module by $m_1 \otimes 1, \dots, m_k \otimes 1$. Thus it is a finite module. □

Corollary 16 Stability under Products.

Let $f : X \to Y$ and $f' : X' \to Y'$ be morphisms having property $\mathcal{P}$. Then the product morphism $f \times f' : X \times X' \to Y \times Y'$ (over $\operatorname{Spec} \mathbb{Z}$ or a common base S) has property $\mathcal{P}$.

Proof. The product morphism can be decomposed as a composition of a base change and a morphism (viewing the product as a fibered product over the terminal object or base S). Consider the diagram:

$$X \times X' \xrightarrow{id \times f'} X \times Y' \xrightarrow{f \times id} Y \times Y'$$

We decompose $f \times f'$ as the composition:

$$(f \times f') = (f \times \mathrm{id}_{Y'}) \circ (\mathrm{id}_X \times f').$$

1. The morphism $f \times \mathrm{id}_{Y'} : X \times Y' \to Y \times Y'$ is obtained by base changing $f : X \to Y$ along the projection $Y \times Y' \to Y$. Since $\mathcal{P}$ is **stable under base change**, this map has property $\mathcal{P}$.
2. The morphism $\mathrm{id}_X \times f' : X \times X' \to X \times Y'$ is obtained by base changing $f' : X' \to Y'$ along the projection $X \times Y' \to Y'$. Since $\mathcal{P}$ is **stable under base change**, this map has property $\mathcal{P}$.
3. Since $\mathcal{P}$ is **stable under composition**, the composition of these two maps has property $\mathcal{P}$.

Thus, the product morphism satisfies $\mathcal{P}$. □

We analyze the stability of three fundamental classes of morphisms:

- **Separated:** (Diagonal is a closed immersion).
- **Proper:** (Separated + Finite Type + Universally Closed).
- **Projective:** (Factors as a closed immersion into projective space followed by projection).

Proposition 17 Stability of Separated Morphisms.

(i) **Base Change:** If $f : X \to Y$ is separated, then for any $Y' \to Y$, the base change $f' : X' \to Y'$ is separated.

(ii) **Composition:** If $f : X \to Y$ and $g : Y \to Z$ are separated, then $g \circ f : X \to Z$ is separated.

(iii) **Products:** If f and g are separated, then $f \times g$ is separated.

Proof. **(i) Base Change:** Let $X' = X \times_Y Y'$. We have a Cartesian diagram connecting the diagonals:

$$\begin{tikzcd}
X' \arrow[r, "\Delta_{X'/Y'}"] \arrow[d] & X' \times_{Y'} X' \arrow[d] \\
X \arrow[r, "\Delta_{X/Y}"] & X \times_Y X
\end{tikzcd}$$

Since f is separated, $\Delta_{X/Y}$ is a closed immersion. We proved in the previous section that **closed immersions are stable under base change**. Thus, $\Delta_{X'/Y'}$ is a closed immersion, so f' is separated.

(ii) Composition: We need to show $\Delta_{X/Z} : X \to X \times_Z X$ is a closed immersion. Consider the factorization of the diagonal $\Delta_{X/Z}$ through the relative product over Y:

$$X \xrightarrow{\Delta_{X/Y}} X \times_Y X \xrightarrow{h} X \times_Z X.$$

The map $\Delta_{X/Y}$ is a closed immersion because f is separated.

The map $h : X \times_Y X \to X \times_Z X$ is the base change of the diagonal $\Delta_{Y/Z} : Y \to Y \times_Z Y$ via the morphism $X \times_Z X \to Y \times_Z Y$ (specifically, $X \times_Y X \cong (X \times_Z X) \times_{(Y \times_Z Y)} Y$).

Since g is separated, $\Delta_{Y/Z}$ is a closed immersion. By stability under base change, h is a closed immersion.

Since $\Delta_{X/Z}$ is the composition of two closed immersions ($h \circ \Delta_{X/Y}$), it is a closed immersion.

(iii) Products: Follows immediately from Composition + Base Change, exactly as proven for general properties in the previous section. □

Proposition 18 Stability of Proper Morphisms.

Proper morphisms are stable under base change, composition, and products.

Proof. Recall that a morphism f is **proper** if it is **separated**, of **finite type**, and **universally closed**. We have already established stability for "Separated" and "Finite Type". We focus on "Universally Closed".

1. Base Change: Let $f : X \to Y$ be universally closed. Let $Y' \to Y$ be a base change. The morphism $f' : X \times_Y Y' \to Y'$ is universally closed by the very definition of "universal": for any $Z \to Y'$, the base change of f' along Z is isomorphic to the base change of f along the composite $Z \to Y' \to Y$, which is closed.

2. Composition: Let $f : X \to Y$ and $g : Y \to Z$ be proper. We need to show $g \circ f$ is universally closed. Let $T \to Z$ be any test morphism. Consider the base change diagram:

$$X_T \xrightarrow{f_T} Y_T \xrightarrow{g_T} T.$$

Since g is universally closed, g_T is a closed map. Since f is universally closed, its base change f_T (along $Y_T \to Y$) is a closed map. The composition of closed maps is closed. Thus $(g \circ f)_T = g_T \circ f_T$ is closed. □

Proposition 19 Stability of Projective Morphisms.

Projective morphisms are stable under base change, composition, and products.

Proof. A morphism $f : X \to Y$ is **projective** if it factors as a closed immersion $X \hookrightarrow \mathbb{P}^n_Y$ followed by the projection $\mathbb{P}^n_Y \to Y$.

1. Base Change: Given $Y' \to Y$, the fiber product $\mathbb{P}^n_Y \times_Y Y'$ is canonically isomorphic to $\mathbb{P}^n_{Y'}$ (projective space commutes with base change). If $X \hookrightarrow \mathbb{P}^n_Y$ is a closed immersion, its base change $X' \to \mathbb{P}^n_{Y'}$ remains a closed immersion. Thus X' is projective over Y'.

2. Composition (The Segre Embedding): Let $f : X \to Y$ and $g : Y \to Z$ be projective. By definition:

- There exists a closed immersion $i : Y \hookrightarrow \mathbb{P}^m_Z$.
- There exists a closed immersion $j : X \hookrightarrow \mathbb{P}^n_Y$.

We want to embed X into some $\mathbb{P}^N_Z$.

Step 1: Lift X to a product over Z. Since $\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y$, we have an embedding $X \hookrightarrow \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y$. Since $Y \hookrightarrow \mathbb{P}^m_Z$ is a closed immersion, we have an induced closed immersion:

$$X \hookrightarrow \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^m_Z \cong \mathbb{P}^n_Z \times_Z \mathbb{P}^m_Z.$$

Step 2: The Segre Embedding. We use the **Segre Embedding** to map the product of projective spaces into a larger projective space. Let $N = (n+1)(m+1) - 1$. We define the map:

$$\Psi : \mathbb{P}^n_Z \times_Z \mathbb{P}^m_Z \longrightarrow \mathbb{P}^N_Z.$$

In terms of homogeneous coordinates, if $[x_0 : \cdots : x_n]$ are coordinates on $\mathbb{P}^n$ and $[y_0 : \cdots : y_m]$ on $\mathbb{P}^m$, the map sends:

$$([x_i], [y_j]) \longmapsto [\cdots : x_i y_j : \dots].$$

It is a standard result of classical algebraic geometry that Ψ is a **closed immersion**.

Conclusion: Composing the closed immersions:

$$X \hookrightarrow \mathbb{P}^n_Z \times_Z \mathbb{P}^m_Z \xrightarrow{\Psi} \mathbb{P}^N_Z.$$

Thus X is a closed subscheme of $\mathbb{P}^N_Z$, which means $g \circ f$ is projective.

3. Products: Follows from Base Change + Composition. □

Proposition 20 Target-Locality of Separated and Proper Morphisms.

Let $f : X \to Y$ be a morphism.

(i) **Separatedness is local on the target.**

(ii) **Properness is local on the target.**

Proof. Let $\{V_i\}$ be an open covering of Y. Let $f_i : f^{-1}(V_i) \to V_i$ be the restrictions.

(i) Separatedness: By definition, f is separated if and only if the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion. We know that the property "**being a closed immersion**" is local on the target (proven in previous section). The target of Δ is $X \times_Y X$. An open cover of Y induces an open cover of the product: $\{f^{-1}(V_i) \times_{V_i} f^{-1}(V_i) \cong (X \times_Y X)|_{V_i}\}$. Over each V_i, the morphism corresponds to the diagonal $\Delta_i : f^{-1}(V_i) \to f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$. Thus, Δ is a closed immersion globally if and only if each Δ_i is a closed immersion locally.

(ii) Properness: Recall that Proper = Separated + Finite Type + Universally Closed. We have already established that **Separatedness** and **Finite Type** are target-local. It remains to show that **Universally Closed** is target-local.

Suppose f_i is universally closed for all i. Let $X' \to Y'$ be any base change of f along $Y' \to Y$. We want to show $f' : X' \to Y'$ is a closed map. Being a closed map is a local property on the target: a subset $Z \subset Y'$ is closed if and only if $Z \cap V'_j$ is closed for an open cover $\{V'_j\}$ of Y'. We can choose a cover of Y' such that each open set maps into some V_i. Thus, we reduce to the case where the base change is along a morphism $Y' \to V_i$. In this case, the base change of f is isomorphic to the base change of f_i. Since f_i is universally closed, its base change is a closed map. Thus, f is universally closed. □

Note: The Definition of Projective Morphisms

The property of being **projective** depends heavily on the definition:

- **Strict/Global Definition (Hartshorne):** $f : X \to Y$ is projective if it factors as a closed immersion $X \hookrightarrow \mathbb{P}^n_Y$ followed by the projection. This property is **NOT local on the target**. Locally on Y, we may have embeddings into projective spaces of different dimensions, or the twisting sheaves $\mathcal{O}(1)$ might not glue to a global ample line bundle (obstructions in $H^1(Y, \mathcal{O}^*)$).
- **EGA Definition (Local):** $f : X \to Y$ is projective if $X \cong \mathsf{Proj}(\mathcal{S})$, where $\mathcal{S}$ is a quasi-coherent graded $\mathcal{O}_Y$-algebra, generated in degree 1 by finitely many sections (finite type). This definition **IS local on the target**, because the construction of Proj and the property of being "generated in degree 1" glue perfectly over an open cover.

In modern scheme theory (and for the cancellation properties below), the EGA definition is often preferred for its flexibility.

Lemma 21 Graph of a Morphism.

Let $f : X \to Y$ be a morphism over Z (i.e., $g : Y \to Z$ is the structure map). The **graph morphism** is defined as:

$$\Gamma_f = (\mathrm{id}_X, f) : X \longrightarrow X \times_Z Y.$$

If $Y \to Z$ is a **separated** morphism, then the graph Γ_f is a **closed immersion**.

Proof. Consider the following Cartesian diagram (pullback of the diagonal):

$$\begin{tikzcd}
X \arrow[r, "\Gamma_f"] \arrow[d, "f"] & X \times_Z Y \arrow[d, "f \times \mathrm{id}_Y"] \\
Y \arrow[r, "\Delta_{Y/Z}"] & Y \times_Z Y
\end{tikzcd}$$

The map $X \times_Z Y \to Y \times_Z Y$ is given by $(x, y) \mapsto (f(x), y)$. The inverse image of the diagonal $\Delta_{Y/Z}(Y) = \{(y, y)\}$ is exactly the set $\{(x, y) \mid f(x) = y\}$, which is the image of the graph Γ_f. Since $Y \to Z$ is separated, $\Delta_{Y/Z}$ is a closed immersion. By stability of closed immersions under base change, Γ_f is a closed immersion. □

Proposition 22 Cancellation Properties.

Let $f : X \to Y$ and $g : Y \to Z$ be morphisms. Let $h = g \circ f$.

(i) **Separatedness (Absolute Cancellation):** If h is separated, then f is separated (for *any* g).

(ii) **Proper/Projective (Relative Cancellation):** If h is proper (resp. projective) and g is **separated**, then f is proper (resp. projective).

Proof. **(i) Cancellation for Separatedness:** Assume $h : X \to Z$ is separated. We want to show $\Delta_{X/Y} : X \to X \times_Y X$ is a closed immersion. There is a canonical immersion $j : X \times_Y X \to X \times_Z X$ (the diagonal map of the fiber product diagram). The diagonal of f factors through the diagonal of h:

$$\Delta_{X/Y} : X \xrightarrow{\Delta_{X/Z}} X \times_Z X \quad \text{(This factorization is set-theoretic/topological).}$$

More precisely, the image $\Delta_{X/Y}(X)$ is exactly the inverse image of the closed set $\Delta_{X/Z}(X)$ under the continuous map j.

$$\Delta_{X/Y}(X) = j^{-1}(\Delta_{X/Z}(X)).$$

Since h is separated, $\Delta_{X/Z}(X)$ is closed in $X \times_Z X$. Since j is continuous, $\Delta_{X/Y}(X)$ is closed in $X \times_Y X$. Thus f is separated.

(ii) Cancellation for Proper/Projective: Assume h is proper (resp. projective) and g is separated. We factor f using its graph over Z:

$$f : X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{\mathrm{pr}_Y} Y.$$

- **Step 1:** Since g is separated, by the **Graph Lemma**, Γ_f is a **closed immersion**.
- **Step 2:** The projection $\mathrm{pr}_Y : X \times_Z Y \to Y$ is the base change of $h : X \to Z$ along $g : Y \to Z$.

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d, "\mathrm{pr}_Y"] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

  Since h is proper (resp. projective), and these properties are **stable under base change**, the projection pr_Y is proper (resp. projective).

- **Step 3:** f is the composition of a closed immersion (Γ_f) and a proper (resp. projective) morphism (pr_Y). Since closed immersions are proper and projective, and these properties are **stable under composition**, f is proper (resp. projective).

□

Proposition 23 Hierarchy of Morphisms.

(1) Any **closed immersion** is a **projective morphism**.

(2) Any **immersion** (open, closed, or locally closed) is a **separated morphism**.

(3) Any **projective morphism** is a **proper morphism**.

Proof. **(1) Closed Immersion $\implies$ Projective** Let $f : X \to Y$ be a closed immersion. We recall the definition of a projective morphism: f is projective if it factors as a closed immersion $i : X \hookrightarrow \mathbb{P}^n_Y$ followed by the projection $\pi : \mathbb{P}^n_Y \to Y$.

Consider the projective space of dimension 0 over Y, denoted $\mathbb{P}^0_Y$. By definition, $\mathbb{P}^0_Y = \mathbb{P}^0_{\mathbb{Z}} \times_{\mathbb{Z}} Y$. Since $\mathbb{P}^0_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}$, the fiber product is isomorphic to Y itself:

$$\mathbb{P}^0_Y \cong Y.$$

The projection map $\pi : \mathbb{P}^0_Y \to Y$ is an isomorphism. We can factor f as:

$$X \xrightarrow{f} Y \xrightarrow{\sim} \mathbb{P}^0_Y \xrightarrow{\pi} Y.$$

The first part $X \to \mathbb{P}^0_Y$ is the composition of the closed immersion f with an isomorphism, hence it is a closed immersion. Thus, f factors as a closed immersion into a projective space followed by the projection. So f is projective.

(2) Immersion $\implies$ Separated Let $f : X \to Y$ be an immersion. By definition, f factors as an open immersion $j : X \to Z$ followed by a closed immersion $i : Z \to Y$. Since we proved that **separatedness is stable under composition**, it suffices to show that open immersions and closed immersions are separated.

However, there is a more direct argument using the categorical property of immersions.

- **Monomorphism:** An immersion $f : X \to Y$ is a **monomorphism** in the category of schemes. Geometrically, this is because f is injective on the underlying topological spaces and surjective on the sheaves of rings (on the stalks).
- **Diagonal Isomorphism:** The diagonal morphism $\Delta : X \to X \times_Y X$ is defined by the universal property of the fiber product. Since f is a monomorphism, the two projections $p_1, p_2 : X \times_Y X \to X$ must be equal (since $f \circ p_1 = f \circ p_2$). This implies that the diagonal map Δ is actually an **isomorphism** between X and $X \times_Y X$.
- **Conclusion:** Since Δ is an isomorphism, and every isomorphism is a closed immersion (it corresponds to the surjective ring map $A \to A$), Δ is a closed immersion.

Thus, f is separated. □

7.2 Valuative Criteria for Separatedness and Properness

To state the criteria, we first fix some notation regarding valuation rings. Let R be a valuation ring of a field K. Let $T = \operatorname{Spec} R$ and $U = \operatorname{Spec} K$. The natural inclusion $R \hookrightarrow K$ induces a morphism $U \to T$. Geometrically, T consists of two points: a generic point $t_1 = (0)$ and a closed point $t_0 = \mathfrak{m}_R$. The morphism $U \to T$ maps the point of U to t_1.

Lemma 24 Geometric Meaning of Valuation Maps.

Let R be a valuation ring of a field K, $T = \operatorname{Spec} R$, $U = \operatorname{Spec} K$. Let X be a scheme.

(i) To give a morphism $U \to X$ is equivalent to giving a point $x_1 \in X$ and an inclusion of fields $k(x_1) \subseteq K$.

(ii) To give a morphism $T \to X$ is equivalent to giving two points $x_0, x_1 \in X$, with x_0 a specialization of x_1 (i.e., $x_0 \in \overline{\{x_1\}}$), and an inclusion of fields $k(x_1) \subseteq K$, such that R dominates the local ring $\mathcal{O}_{X,x_0}$ on the subscheme $Z = \overline{\{x_1\}}$ (with reduced induced structure).

Proof. **(i)** U is a one-point scheme with structure sheaf K. A morphism $U \to X$ maps the point to x_1 and induces a local homomorphism $\mathcal{O}_{X,x_1} \to K$. This factorizes through the residue field $k(x_1)$, giving the inclusion $k(x_1) \subseteq K$.

(ii) Let t_0, t_1 be the closed and generic points of T. Given $T \to X$, let x_0, x_1 be their images. Since $t_0 \in \overline{\{t_1\}}$, continuity implies $x_0 \in \overline{\{x_1\}}$. Furthermore, the map on local rings gives $\mathcal{O}_{X,x_0} \to \mathcal{O}_{T,t_0} = R$. This map factors through the local ring of x_0 on the closure Z, denoted $\mathcal{O}_{Z,x_0}$. Since R is a valuation ring of K (which contains $k(x_1)$), the condition that the map is local means R dominates $\mathcal{O}_{Z,x_0}$. Conversely, given such data, the dominance relation $\mathcal{O}_{Z,x_0} \to R$ induces the morphism $T \to X$. □

Theorem 25 Valuative Criterion for Separatedness.

Let $f : X \to Y$ be a morphism of **Noetherian** schemes. Then f is **separated** if and only if the following condition holds:

For any field K and any valuation ring R of K (let $T = \operatorname{Spec} R, U = \operatorname{Spec} K$), and for any commutative diagram:

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T \arrow[r] & Y \end{tikzcd}$$

there is **at most one** morphism $T \to X$ making the diagram commute.

Proof. **(⇒) Suppose f is separated.** Assume there are two morphisms $h, h' : T \to X$ making the diagram commute. By the universal property of the fiber product, these induce a single morphism $h'' : T \to X \times_Y X$. Since $h|_U = h'|_U$ (given by the top arrow $U \to X$), the generic point t_1 of T maps to the diagonal $\Delta(X) \subset X \times_Y X$. Since f is separated, the diagonal $\Delta(X)$ is a **closed** subset. The closed point t_0 is a specialization of t_1, so its image under h'' must lie in the closure of the image of t_1. Thus $h''(t_0) \in \Delta(X)$. This means $h(t_0) = h'(t_0) = x_0$. Furthermore, the maps on local rings induced by h and h' must coincide because they coincide on the fraction field level and $X \to Y$ is separated (algebraically, extensions of maps into separated schemes are unique). Thus $h = h'$.

(⇐) Suppose the condition holds. We use the diagonal characterization: f is separated iff $\Delta(X)$ is closed in $X \times_Y X$. Since X is Noetherian, Δ is quasi-compact. It suffices to show that $\Delta(X)$ is **stable under specialization**. Let $\xi_1 \in \Delta(X)$ and let $\xi_1 \rightsquigarrow \xi_0$ be a specialization in $X \times_Y X$. We must show $\xi_0 \in \Delta(X)$. Let $K = k(\xi_1)$. Let $\mathcal{O}$ be the local ring of ξ_0 on the subscheme $\overline{\{\xi_1\}}$ (with reduced induced structure). By domination theory (algebraic fact), there exists a valuation ring R of K dominating $\mathcal{O}$. This gives a morphism $T = \operatorname{Spec} R \to X \times_Y X$ sending $t_1 \to \xi_1$ and $t_0 \to \xi_0$. Composing with projections p_1, p_2, we get two morphisms $h, h' : T \to X$. Since $\xi_1 \in \Delta(X)$, we have $p_1(\xi_1) = p_2(\xi_1)$, so h and h' agree on the generic point (i.e., on U). By the uniqueness hypothesis of the theorem, we must have $h = h'$. Therefore, the morphism $T \to X \times_Y X$ factors through the diagonal, implying $\xi_0 \in \Delta(X)$. □

To prove properness, we need a lemma connecting "closed maps" with "specialization".

Lemma 26 Specialization and Closed Maps.

Let $f : X \to Y$ be a **quasi-compact** morphism of schemes. Then the subset $f(X)$ is closed in Y if and only if it is **stable under specialization**.

Proof. One direction is obvious (closed sets are stable under specialization). For the converse, assume $f(X)$ is stable under specialization. We can assume Y is affine and reduced (by taking closure and reduced structure), and X is a finite union of affine opens X_i (by quasi-compactness). Let $y \in \overline{f(X)}$. We want to show $y \in f(X)$. Since $\overline{f(X)} = \bigcup \overline{f(X_i)}$, $y \in \overline{f(X_i)}$ for some i. We reduce to the case of a dominant morphism of affine schemes $X_i = \operatorname{Spec} A \to Y_i = \operatorname{Spec} B$ where $B \to A$ is injective. The point y corresponds to a prime $\mathfrak{p} \subset B$. Let $\mathfrak{p}' \subseteq \mathfrak{p}$ be a minimal prime of B. $\mathfrak{p}'$ corresponds to a point y' specializing to y. Since $B \to A$ is injective, minimal primes of B lift to minimal primes of A (going-down for flat/free, or by tensor product argument over the field of fractions). Specifically, we can find a point $x' \in X_i$ mapping to y'. Thus $y' \in f(X)$. Since $f(X)$ is stable under specialization and $y' \rightsquigarrow y$, we have $y \in f(X)$. □

Theorem 27 Valuative Criterion for Properness.

Let $f : X \to Y$ be a morphism of **finite type** with X **Noetherian**. Then f is **proper** if and only if for every valuation ring R and diagram:

$$\begin{tikzcd}
U \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
T \arrow[r] \arrow[ur, dashed, "\exists!"] & Y
\end{tikzcd}$$

there exists a **unique** morphism $T \to X$ making the diagram commute.

Proof. **(⟹) Existence part (Uniqueness follows from Separatedness).** Suppose f is proper. By definition, it is separated, so uniqueness holds by Theorem 25. We consider the base extension $f_T : X_T \to T$ where $X_T = X \times_Y T$. The given maps $U \to X$ and $U \to T$ induce a map $U \to X_T$ (a section over the generic point). Let ξ_1 be the image of t_1 in X_T. Let $Z = \overline{\{\xi_1\}}$ with reduced structure. Since f is proper, it is universally closed. Thus $f_T : X_T \to T$ is a closed map. So $f_T(Z)$ is a closed subset of T. Since it contains the generic point t_1, and T is local, $f_T(Z) = T$. Thus, there exists a point $\xi_0 \in Z$ mapping to the closed point $t_0 \in T$. This implies we have a local homomorphism of local rings $R \to \mathcal{O}_{Z,\xi_0}$. Since R is a valuation ring, it is maximal for the relation of domination, so $R \cong \mathcal{O}_{Z,\xi_0}$. This isomorphism yields the section $T \to X_T$, which projects to the desired map $T \to X$.

(⟸) Converse. Suppose the condition holds.

1. **Separated:** Uniqueness implies f is separated (Theorem 25).

2. **Finite Type:** Given by hypothesis.

3. **Universally Closed:** We must show that for any $Y' \to Y$, the base change $f' : X' \to Y'$ is closed. Since f is finite type, so is f'. It suffices to show that for any closed $Z \subseteq X'$, the image $f'(Z)$ is closed. By **Lemma 26**, it is enough to show $f'(Z)$ is stable under specialization. Let $z_1 \in Z$, $y_1 = f'(z_1)$, and let $y_1 \rightsquigarrow y_0$ be a specialization in Y'. Let R be a valuation ring in the field $K = k(z_1)$ dominating the local ring of y_0 on $\overline{\{y_1\}}$. This gives maps $U \to Z \hookrightarrow X'$ and $T \to Y'$. Composing with $X' \to X$ and $Y' \to Y$, we get a diagram for $f : X \to Y$. By hypothesis, there exists a unique lift $T \to X$. Since X' is a fiber product, this lifts to a map $T \to X'$ (diagram chase). Because Z is closed and the generic point maps to Z, the whole image of T lies in Z. Thus the closed point maps to some $z_0 \in Z$. By commutativity, $f'(z_0) = y_0$. Thus $y_0 \in f'(Z)$.

Therefore f is universally closed, hence proper. □

Corollary 28 Projective Implies Proper.

Let S be a scheme. The projective n-space $\mathbb{P}^n_S$ is proper over S. Consequently, any projective S-scheme is proper over S.

Proof. A morphism $f : X \to S$ is **proper** if it is separated, of finite type, and universally closed. Using the **Valuative Criterion of Properness**, for a Noetherian scheme (or more generally, using the general form), f is proper if and only if for every Discrete Valuation Ring (DVR) R with fraction field K, and every commutative diagram:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\varphi_K"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r, "\psi"] \arrow[ur, dashed, "\exists! \widetilde{\varphi}"] & S
\end{tikzcd}$$

there exists a unique morphism $\widetilde{\varphi} : \operatorname{Spec} R \to X$ making the diagram commute.

Step 1: Separatedness (Uniqueness) The separatedness of $\mathbb{P}^n_S \to S$ is a standard result (the diagonal is a closed immersion). In the context of the valuative criterion, this corresponds to the uniqueness of the lift $\widetilde{\varphi}$. Since $\mathbb{P}^n_S$ is covered by affine charts $U_i \cong \mathbb{A}^n_S$ which are separated, and the valuation is determined by its value on functions, the extension from K to R is unique if it exists.

Step 2: Universal Closure (Existence) We focus on the existence of the morphism $\widetilde{\varphi} : \operatorname{Spec} R \to \mathbb{P}^n_S$.

A morphism $\operatorname{Spec} K \to \mathbb{P}^n_S$ corresponds to a K-valued point of $\mathbb{P}^n$. Algebraically, this is given by a set of elements $(x_0, x_1, \ldots, x_n)$ in K, where not all x_i are zero. These are the homogeneous coordinates of the point in $\mathbb{P}^n(K)$.

Let $v : K^\times \to \mathbb{Z}$ be the valuation associated with the DVR R. Let π be a uniformizer of R. Consider the set of valuations of the coordinates: $\{v(x_0), v(x_1), \ldots, v(x_n)\}$. Since there are finitely many coordinates, there exists an index j such that:

$$m = \min_{0 \le i \le n} \{v(x_i)\} = v(x_j)$$

Define new coordinates $x_i' \in K$ by multiplying each x_i by π^{-m}:

$$x_i' = \pi^{-m} x_i \quad \text{for } i = 0, \ldots, n$$

By the definition of m, we have:

$$v(x_i') = v(\pi^{-m}) + v(x_i) = -m + v(x_i) \ge 0$$

Therefore, $x_i' \in R$ for all i. Furthermore, for the index j where the minimum was attained:

$$v(x_j') = -m + m = 0$$

This implies that x_j' is a **unit** in R ($x_j' \in R^\times$).

The tuple $(x_0', x_1', \ldots, x_n')$ defines a morphism $\widetilde{\varphi} : \operatorname{Spec} R \to \mathbb{P}^n_R \to \mathbb{P}^n_S$. Specifically, since at least one coordinate (x_j') is a unit, the point does not vanish modulo the maximal ideal $\mathfrak{m} \subset R$. Thus, it defines a well-defined R-valued point.

Step 3: Conclusion for Projective Schemes A general projective scheme X is, by definition, a closed subscheme of some $\mathbb{P}^n_S$.

- We have shown $\mathbb{P}^n_S \to S$ is proper.
- Closed immersions are proper.
- The composition of proper morphisms is proper.

Thus, $X \hookrightarrow \mathbb{P}^n_S \to S$ is proper. □ □

7.3 Chow's Lemma

Intuition 29 Chow's Lemma Intuition.

In this section, we will focus all our efforts on proving an extremely important and renowned lemma in algebraic geometry: Chow's Lemma.

Chow's Lemma is named after the great Chinese mathematician and algebraic geometer, Wei-Liang Chow (1911-1995). Mr. Chow was one of the principal figures in modern algebraic geometry; the majority of his work was pioneering, establishing both the foundations and powerful tools for the theory of algebraic geometry. Chow notably collaborated with Kunihiko Kodaira and invited him to lecture at Johns Hopkins University.

Chow's Lemma establishes a crucial connection between proper morphisms and projective morphisms, allowing many desirable properties that hold for projective morphisms to be generalized to the broader class of proper morphisms. Since the proof of this lemma is lengthy and technical, we will divide it into several parts.

We now prove one of the most technical yet fundamental results in the theory of schemes. Chow's Lemma asserts that while proper morphisms are not always projective, they are "birationally" projective. This allows us to reduce many questions about proper schemes to the projective case.

Theorem 30 Chow's Lemma.

Let S be a Noetherian scheme and let $f : X \to S$ be a **proper** morphism. Then there exists a scheme X' and a morphism $g : X' \to X$ such that:

1. g is a **projective** morphism.
2. The composite $f \circ g : X' \to S$ is a **projective** morphism.
3. There exists a dense open subset $U \subseteq X$ such that $g^{-1}(U)$ is dense in X', and the restriction $g|_{g^{-1}(U)} : g^{-1}(U) \to U$ is an isomorphism.

In short, f is dominated by a projective morphism $f \circ g$ that is an isomorphism over a dense open set.

The proof is long, so we divide it into three strategic steps: reduction to the irreducible case, local projective approximation, and the global graph construction.

Step 1: Reduction to the Irreducible Case. Since S is Noetherian and f is of finite type (being proper), X is a Noetherian scheme. It has finitely many irreducible components $X_1, \dots, X_n$. Let X_i be endowed with the reduced induced closed subscheme structure. Suppose we can prove the lemma for each restricted morphism $f_i : X_i \to S$. That is, for each i, there exists a projective $g_i : X_i' \to X_i$ with the required properties over a dense open $U_i \subseteq X_i$. We can define X' as the disjoint union $X' := \coprod_{i=1}^n X_i'$ and $g = \coprod g_i$.

- i. g is projective because a disjoint union of projective morphisms is projective.
- ii. X' is birational to X because the union of the dense open sets U_i (disjoint from intersections $X_i \cap X_j$) forms a dense open set in X.

Thus, we may assume without loss of generality that X is irreducible.

Step 2: Local Projective Factorization. Since X is of finite type over S, we can cover X by finitely many affine open subschemes U_i ($i = 1, \dots, m$) such that each $U_i \to S$ factors through an immersion into a projective space. Specifically, since U_i is affine of finite type over S, there is a closed immersion $U_i \hookrightarrow \mathbb{A}^{n_i}_S \hookrightarrow \mathbb{P}^{n_i}_S$. Let $P_i = \mathbb{P}^{n_i}_S$. We have an immersion $j_i : U_i \to P_i$ over S. By 20, we can factor j_i as:

$$U_i \xrightarrow{\text{open}} \overline{U_i} \xrightarrow{\text{closed}} P_i$$

where $\overline{U_i}$ is the scheme-theoretic image of U_i in P_i. Since $P_i \to S$ is proper (it is projective), the closed subscheme $\overline{U_i}$ is also proper over S. Let $Y_i = \overline{U_i}$. We have constructed a diagram:

$$\begin{tikzcd}
U_i \arrow[r, hook, "open"] \arrow[d, hook] & Y_i \arrow[d, "proper"] \\
X \arrow[r, "proper"] & S
\end{tikzcd}$$

Note that the map $U_i \to Y_i$ is an open immersion, and $Y_i \to S$ is projective. However, these Y_i are separate. We need to "glue" them. But we cannot glue them directly; instead, we will use the "graph trick" to embed the common part into the product.

Let $U = \bigcap_{i=1}^{m} U_i$. Since X is irreducible, the intersection of non-empty open sets is non-empty and **dense** in X. This U will be our "dense open set" for the final isomorphism.

Step 3: Global Construction via the Graph. We glue the local approximations using a product construction. Let $P = P_1 \times_S P_2 \times_S \cdots \times_S P_m$. Since each $P_i \to S$ is projective, their product $P \to S$ is projective (via the Segre embedding). Consider the product scheme:

$$Z := X \times_S P.$$

We define a morphism $\psi : U \to Z$ on the dense open set $U = \bigcap U_i$. The components of ψ are:

i. The inclusion $j : U \hookrightarrow X$.

ii. The maps $\phi_k : U \hookrightarrow U_k \to P_k$ for each $k = 1, \ldots, m$.

Let X' be the **scheme-theoretic image** (closure) of ψ in Z. We have a diagram:

$$\begin{tikzcd}
U \arrow[r, hook, "\psi"] \arrow[dr, "j"'] & X' \arrow[r, hook, "h"] \arrow[d, "g"] & X \times_S P \arrow[r, "\pi_2"] \arrow[dl, "\pi_1"] & P \arrow[d] \\
 & X \arrow[r, "f"] & S & S
\end{tikzcd}$$

where $g = \pi_1|_{X'}$ and $h = \pi_2|_{X'}$.

Step 4: Verification of Properties. *1. g is an isomorphism over U.* The restriction $g^{-1}(U)$ is the closure of the graph of ψ restricted to the fiber over U. Since the graph over U is isomorphic to U, g is an isomorphism over U.

2. g is projective. g is the composition of the closed immersion $X' \hookrightarrow X \times_S P$ and the projection $X \times_S P \to X$. Since $P \to S$ is projective, the base change $X \times_S P \to X$ is projective. A closed subscheme of a projective scheme is projective. Thus g is projective.

3. $f \circ g$ is projective. It suffices to show that the map $h : X' \to P$ is a **closed immersion**. If so, X' is isomorphic to a closed subscheme of P, and since $P \to S$ is projective, $X' \to S$ is projective.

We use the criterion: **Proper + Immersion $\implies$ Closed Immersion.**

i. h **is Proper:** Factor h as $X' \hookrightarrow X \times_S P \xrightarrow{\pi_2} P$. The first map is a closed immersion. The second map π_2 is the base change of $f : X \to S$. Since f is **proper** (hypothesis), π_2 is proper. Thus h is proper.

ii. h **is an Immersion:** Being an immersion is local on the target. We cover the target P by open sets to check this. Recall $U_k \to P_k$ is an immersion. Let $V_k \subseteq P_k$ be an open set such that $U_k \to V_k$ is a closed immersion. Let $\mathcal{W}_k = \pi_k^{-1}(V_k) \subseteq P$ be the inverse image in the product. These open sets $\{\mathcal{W}_k\}_{k=1}^{m}$ cover the image of h. Consider the restriction of h over one such open set $\mathcal{W}_k$:
$$h_k : h^{-1}(\mathcal{W}_k) \longrightarrow \mathcal{W}_k.$$

Over this open set, the map factors through the behavior of the k-th component. Specifically, on $g^{-1}(U_k)$, the map $X' \to P$ acts like the graph of the map $U_k \to P$. The morphism h restricted to the preimage of $\mathcal{W}_k$ identifies $X' \cap (X \times \mathcal{W}_k)$ with the **graph** of a morphism to the other factors, composed with the closed immersion $U_k \to V_k$. Since the graph of a morphism to a separated scheme is a closed immersion, and $U_k \to V_k$ is a closed immersion, the composition is an immersion. Since h is locally an immersion on the target, it is globally an immersion.

Since h is a proper morphism and an immersion, it must be a **closed immersion**. Thus X' is projective over S.

□

The power of Chow's Lemma lies in its ability to transfer "finiteness properties" from projective schemes (where we have ample line bundles and explicit coordinates) to arbitrary proper schemes.

We now address the question raised before: *When does the direct image functor f_* preserve coherence?*

Theorem 31 Direct Image Theorem for Proper Morphisms.

Let S be a Noetherian scheme and let $f : X \to S$ be a **proper** morphism. If $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module, then the direct image $f_*\mathscr{F}$ is a coherent $\mathscr{O}_S$-module.

Proof. We proceed by induction on the dimension of the support of $\mathscr{F}$. The case where $\mathscr{F}$ is supported on a finite set (dimension 0) is trivial (finite morphism). Assume the theorem holds for all coherent sheaves supported on closed subschemes of dimension $< d$. Let $\dim(\mathrm{supp}(\mathscr{F})) = d$.

Step 1: Introduction of Projective Cover. Apply Chow's Lemma to the proper morphism $X \to S$. There exists a projective morphism $\pi : X' \to X$ which is an isomorphism over a dense open subset $U \subseteq X$. Since π is projective, the composite $f \circ \pi : X' \to S$ is projective. We know (from the theory of projective morphisms, specifically Serre's Theorems) that the direct image theorem holds for **projective** morphisms. Thus $(f \circ \pi)_*(\pi^*\mathscr{F})$ is coherent.

Step 2: Comparison via Adjunction. Consider the adjunction map $\eta : \mathscr{F} \to \pi_*\pi^*\mathscr{F}$. Let $\mathcal{K} = \ker(\eta)$ and $C = \mathrm{coker}(\eta)$. We have an exact sequence of sheaves on X:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathscr{F} \longrightarrow \pi_*\pi^*\mathscr{F} \longrightarrow C \longrightarrow 0.$$

Since π is an isomorphism over the dense open set U, the map η is an isomorphism over U. Therefore, the supports of the kernel $\mathcal{K}$ and cokernel C are contained in $X \setminus U$. Since U is dense, $X \setminus U$ has strictly smaller dimension than X. By the **induction hypothesis**, $f_*\mathcal{K}$ and f_*C are coherent $\mathscr{O}_S$-modules.

Step 3: The Sandwich Logic. We apply the left-exact functor f_* to the sequence. Although f_* is not right exact, the "defect" is controlled by higher cohomology (specifically $R^1 f_*$). If we assume (for the sake of this sketch) that we can handle the exactness, roughly speaking, $f_*\mathscr{F}$ is "sandwiched" between $f_*\mathcal{K}$ (coherent by induction) and $f_*(\pi_*\pi^*\mathscr{F})$. Note that $f_*(\pi_*\pi^*\mathscr{F}) = (f \circ \pi)_*(\pi^*\mathscr{F})$. Since $f \circ \pi$ is **projective**, this term is coherent. Since coherence is closed under kernels, cokernels, and extensions (in the abelian category of coherent sheaves), $f_*\mathscr{F}$ must be coherent. □

The most concrete manifestation of this theorem is when the base $S = \operatorname{Spec} k$.

Corollary 32 Finiteness of Global Sections on Complete Varieties.

Let k be a field and let X be a **complete** algebraic variety over k (i.e., a scheme proper over k). Then for any coherent sheaf $\mathscr{F}$ on X (e.g., the structure sheaf $\mathscr{O}_X$ or a vector bundle), the space of global sections is finite-dimensional:

$$\dim_k \Gamma(X, \mathscr{F}) < \infty.$$

Proof. The structure morphism $f : X \to \operatorname{Spec} k$ is proper. By the theorem, $f_*\mathscr{F}$ is a coherent sheaf on $\operatorname{Spec} k$. By the equivalence of categories, a coherent sheaf on $\operatorname{Spec} k$ is simply a finitely generated k-module, i.e., a finite-dimensional vector space. Since $\Gamma(\operatorname{Spec} k, f_*\mathscr{F}) = \Gamma(X, \mathscr{F})$, the result follows. □

Note 33 Analogy with Complex Analysis.

This result fails spectacularly for non-proper schemes. For example, on the affine line $\mathbb{A}^1_k$, $\Gamma(\mathbb{A}^1, \mathscr{O}) = k[t]$, which is infinite-dimensional. Properness (compactness) forces the algebraic "functions" to be constrained, much like Liouville's Theorem in complex analysis forces bounded holomorphic functions on compact domains to be constant.

CHAPTER 8

Projective Schemes and Their Cohomology

8.1 Projective Schemes

We have constructed the projective space $\mathbb{P}^n_A$ by gluing affine charts. To study more general projective schemes, we need the language of graded rings and homogeneous ideals.

Definition 1 Graded Ring.

A **graded ring** S is a ring equipped with a direct sum decomposition as an additive group:

$$S = \bigoplus_{d=0}^{\infty} S_d,$$

such that multiplication respects the grading: $S_d \cdot S_e \subseteq S_{d+e}$ for all $d, e \geq 0$. Elements in a specific component S_d are called **homogeneous elements of degree** d.

Note 2 The Unit.

If $1 = \sum e_i$ is the decomposition of the identity, then for any homogeneous $x \in S_d$, $x = 1 \cdot x = \sum e_i x$. Since $e_i x \in S_{i+d}$, uniqueness of the direct sum implies $e_0 x = x$ and $e_i x = 0$ for $i > 0$. Since this holds for all homogeneous x, it holds for all x, so $e_0 = 1 \in S_0$. Thus, S_0 is a subring of S, and S is an S_0-algebra.

Definition 3 Homogeneous Ideal.

An ideal $\mathfrak{a} \subseteq S$ is called **homogeneous** (or a graded ideal) if it satisfies any of the following equivalent conditions:

1. $\mathfrak{a} = \bigoplus_{d=0}^{\infty} (\mathfrak{a} \cap S_d)$.
2. If $f \in \mathfrak{a}$ and $f = \sum f_d$ is its homogeneous decomposition, then each component $f_d \in \mathfrak{a}$.
3. $\mathfrak{a}$ is generated by homogeneous elements.

We provide the necessary verification that the definition of homogeneous ideals is consistent and that quotients by such ideals inherit a graded structure.

Proof. Equivalence of Definitions of Homogeneous Ideals We prove the implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (2) $\implies$ (1).

(1) $\implies$ (2): Assume $\mathfrak{a} = \bigoplus(\mathfrak{a} \cap S_d)$. Let $f \in \mathfrak{a}$. Since $\mathfrak{a} \subseteq S$, we can uniquely write $f = \sum f_d$ where $f_d \in S_d$. On the other hand, since $f \in \bigoplus(\mathfrak{a} \cap S_d)$, f can be uniquely written as a sum of elements $g_d \in \mathfrak{a} \cap S_d \subseteq S_d$. By the uniqueness of the direct sum decomposition in S, we must have $f_d = g_d$. Thus $f_d \in \mathfrak{a} \cap S_d \subseteq \mathfrak{a}$.

(2) $\implies$ (3): If every element in $\mathfrak{a}$ breaks down into homogeneous components inside $\mathfrak{a}$, then $\mathfrak{a}$ is generated by the set of all its homogeneous elements:

$$\mathfrak{a} = \langle \{h \in \mathfrak{a} \mid h \text{ is homogeneous}\} \rangle.$$

(3) $\implies$ (2): Suppose $\mathfrak{a}$ is generated by a set of homogeneous elements $\{h_i\}_{i \in I}$, with $d_i = \deg(h_i)$. Let $f \in \mathfrak{a}$. Then we can write $f = \sum_{j=1}^{k} r_j h_{i_j}$ for some $r_j \in S$. Decompose each coefficient r_j into homogeneous components: $r_j = \sum_k (r_j)_k$. Then

$$f = \sum_{j=1}^{k} \left(\sum_l (r_j)_l \right) h_{i_j} = \sum_{j,l} (r_j)_l h_{i_j}.$$

The term $(r_j)_l h_{i_j}$ is homogeneous of degree $l + d_{i_j}$. Since $h_{i_j} \in \mathfrak{a}$, the product is in $\mathfrak{a}$. The homogeneous component f_d of f of degree d is simply the sum of all terms $(r_j)_l h_{i_j}$ such that $l + d_{i_j} = d$. Since each such term is in $\mathfrak{a}$ (being a multiple of a generator), their sum f_d is in $\mathfrak{a}$.

(2) $\implies$ (1): Clearly $\bigoplus(\mathfrak{a} \cap S_d) \subseteq \mathfrak{a}$. For the reverse, let $f \in \mathfrak{a}$. By (2), $f = \sum f_d$ with $f_d \in \mathfrak{a}$. Since $f_d \in S_d$, we have $f_d \in \mathfrak{a} \cap S_d$. Thus $f \in \bigoplus(\mathfrak{a} \cap S_d)$. □

Proposition 4 Quotient by Homogeneous Ideal is Graded.

If $\mathfrak{a}$ is a homogeneous ideal, the quotient ring $S/\mathfrak{a}$ inherits a natural grading:

$$(S/\mathfrak{a})_d \cong S_d/(\mathfrak{a} \cap S_d).$$

Proof. Let $\pi : S \to S/\mathfrak{a}$ be the canonical projection homomorphism. Let $\bar{S} = S/\mathfrak{a}$. We define $\bar{S}_d := \pi(S_d)$. This is the image of the additive group S_d in the quotient. Clearly, $\bar{S} = \pi(\sum S_d) = \sum \bar{S}_d$. We must show this sum is direct.

Suppose we have a sum $\sum_d \bar{x}_d = 0$ in $\bar{S}$, where $\bar{x}_d \in \bar{S}_d$. We lift each $\bar{x}_d$ to an element $x_d \in S_d$.

The equation $\sum \bar{x}_d = 0$ in $S/\mathfrak{a}$ means:

$$\sum_d x_d \in \mathfrak{a}.$$

Let $f = \sum x_d$. Since $\mathfrak{a}$ is a homogeneous ideal, condition of definition implies that each homogeneous component of f must belong to $\mathfrak{a}$. The homogeneous component of degree d of f is exactly x_d. Therefore, $x_d \in \mathfrak{a}$ for all d.

This implies $\pi(x_d) = 0$, i.e., $\bar{x}_d = 0$ for all d. Thus, the decomposition is unique, and

$\bar{S} = \bigoplus_{d \geq 0} \bar{S}_d$.

Finally, the multiplication maps $S_d \times S_e \to S_{d+e}$ descend to $\bar{S}_d \times \bar{S}_e \to \bar{S}_{d+e}$ because π is a ring homomorphism.

For the specific form of the component, consider the restriction of π to S_d:

$$\pi|_{S_d} : S_d \longrightarrow \bar{S}_d.$$

This is surjective by definition. Its kernel is $\{x \in S_d \mid x \in \mathfrak{a}\} = \mathfrak{a} \cap S_d$.

By the First Isomorphism Theorem for modules, we have:

$$\bar{S}_d \cong S_d/(\mathfrak{a} \cap S_d).$$

□

The points of our projective schemes will be homogeneous prime ideals. We need a criterion for primality that only checks homogeneous elements.

Lemma 5 Homogeneous Primality Test.

Let $\mathfrak{p} \subseteq S$ be a homogeneous ideal. Then $\mathfrak{p}$ is a prime ideal if and only if for every pair of **homogeneous** elements $f, g \in S$:

$$fg \in \mathfrak{p} \implies f \in \mathfrak{p} \text{ or } g \in \mathfrak{p}.$$

Proof. ($\Rightarrow$) Obvious. ($\Leftarrow$) Suppose the condition holds for homogeneous elements. Let $f, g \in S$ be general elements such that $fg \in \mathfrak{p}$ but $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$. Write $f = f_0 + \cdots + f_m$ and $g = g_0 + \cdots + g_n$ with $f_m \neq 0, g_n \neq 0$ being the highest degree terms. We proceed by induction on the number of terms.

If we can show that the highest terms f_m and g_n must be in $\mathfrak{p}$, we can subtract them and reduce to a smaller case. Consider the highest degree term of the product fg. It is $f_m g_n$ (degree $m + n$). Since $\mathfrak{p}$ is homogeneous, $fg \in \mathfrak{p}$ implies every homogeneous component of fg is in $\mathfrak{p}$. Thus $f_m g_n \in \mathfrak{p}$.

By our hypothesis on homogeneous elements, this implies $f_m \in \mathfrak{p}$ or $g_n \in \mathfrak{p}$. However, we cannot simply assume $f_m \in \mathfrak{p}$ contradicts $f \notin \mathfrak{p}$. Let's refine the argument: Let f be an element with the *minimal* number of homogeneous components such that there exists g with $fg \in \mathfrak{p}$ but $f, g \notin \mathfrak{p}$. Let f_i be the lowest degree component of f not in $\mathfrak{p}$, and g_j the lowest of g not in $\mathfrak{p}$. Look at the component of degree $i + j$ in fg:

$$(fg)_{i+j} = f_i g_j + \sum_{k<i} f_k g_{i+j-k} + \sum_{k>i} f_k g_{i+j-k}.$$

Since $fg \in \mathfrak{p}$, $(fg)_{i+j} \in \mathfrak{p}$. The terms in the sums involve either f_k with $k < i$ (in $\mathfrak{p}$ by choice of i) or g_l with $l < j$ (in $\mathfrak{p}$ by choice of j). Thus $f_i g_j \in \mathfrak{p}$. By hypothesis, $f_i \in \mathfrak{p}$ or $g_j \in \mathfrak{p}$, a contradiction. □

In affine geometry, the maximal ideal $(x_1, \ldots, x_n)$ corresponds to the origin. In projective geometry, the origin 0 is removed (we construct lines *through* the origin). Thus, this ideal defines the empty set.

Definition 6 Irrelevant Ideal.

The ideal $S_+ := \bigoplus_{d>0} S_d$ is called the **irrelevant ideal**. We define the set Proj S as:

$$\operatorname{Proj} S := \{\mathfrak{p} \in \operatorname{Spec} S \mid \mathfrak{p} \text{ is homogeneous and } S_+ \not\subseteq \mathfrak{p}\}.$$

Proposition 7 Homogeneous Ideals.

Homogeneous ideals behave well under standard operations. If $\mathfrak{a}, \mathfrak{b}$ are homogeneous:

i. Sum $\mathfrak{a} + \mathfrak{b}$, Product $\mathfrak{ab}$, Intersection $\mathfrak{a} \cap \mathfrak{b}$ are homogeneous.

ii. The radical $\sqrt{\mathfrak{a}}$ is homogeneous.

Proof. Let $\mathfrak{a}, \mathfrak{b}$ be homogeneous ideals

Sum and Product: The sum $\mathfrak{a} + \mathfrak{b}$ is generated by the union of the homogeneous generators of $\mathfrak{a}$ and $\mathfrak{b}$, so it is homogeneous. The product $\mathfrak{ab}$ is generated by products xy where x is a homogeneous generator of $\mathfrak{a}$ and y is a homogeneous generator of $\mathfrak{b}$. Since the product of homogeneous elements is homogeneous, $\mathfrak{ab}$ is homogeneous.

Intersection: Let $x \in \mathfrak{a} \cap \mathfrak{b}$. Write $x = \sum x_d$ (homogeneous decomposition). Since $x \in \mathfrak{a}$ and $\mathfrak{a}$ is homogeneous, each $x_d \in \mathfrak{a}$. Similarly, each $x_d \in \mathfrak{b}$. Thus $x_d \in \mathfrak{a} \cap \mathfrak{b}$ for all d. By definition, $\mathfrak{a} \cap \mathfrak{b}$ is homogeneous.

Radical: Let $x \in \sqrt{\mathfrak{a}}$. We want to show its homogeneous components are in $\sqrt{\mathfrak{a}}$. This is non-trivial. Let $x = x_s + \cdots + x_r$ where x_s is the lowest degree term (s) and x_r is the highest (r). We know $x^n \in \mathfrak{a}$ for some n. The highest degree term of x^n is $(x_r)^n$. Since $\mathfrak{a}$ is homogeneous, $(x_r)^n \in \mathfrak{a}$. Thus $x_r \in \sqrt{\mathfrak{a}}$. Since $\sqrt{\mathfrak{a}}$ is an ideal, $x - x_r \in \sqrt{\mathfrak{a}}$. Now $x - x_r = x_s + \cdots + x_{r-1}$. By induction on the number of terms, all components belong to $\sqrt{\mathfrak{a}}$. □

Definition 8 Proj.

As a set, Proj S consists of all homogeneous prime ideals $\mathfrak{p}$ of S such that $S_+ \not\subseteq \mathfrak{p}$.

We equip Proj S with the Zariski topology.

i. **Closed Sets:** For any homogeneous ideal $\mathfrak{a}$, define $V_+(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{a} \subseteq \mathfrak{p}\}$.

ii. **Basic Open Sets:** For any homogeneous $f \in S_+$, define

$$D_+(f) := \operatorname{Proj} S \setminus V_+((f)) = \{\mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p}\}.$$

It is a standard verification (similar to the affine case) that $\{D_+(f)\}$ forms a basis for the topology.

Instead of gluing stalks, we define the structure sheaf directly on the basis $D_+(f)$ using the algebraic operation of "degree-zero localization".

Definition 9 Degree-Zero Localization.

Let $f \in S$ be homogeneous of degree $d > 0$. The localization S_f is a $\mathbb{Z}$-graded ring

$(\deg(s/f^n) = \deg s - nd)$. We define the subring of elements of degree 0:

$$S_{(f)} := (S_f)_0 = \left\{ \frac{s}{f^n} \in S_f \,\middle|\, \deg s = n \cdot \deg f \right\}.$$

Definition 10 Structure Sheaf on Proj.

We define a presheaf $\mathscr{O}_{\operatorname{Proj} S}$ on the basis $\{D_+(f)\}$ by:

$$\mathscr{O}_{\operatorname{Proj} S}(D_+(f)) := S_{(f)}.$$

For $D_+(g) \subseteq D_+(f)$, which implies $g^k = fh$ for some homogeneous h, there is a natural map $S_{(f)} \to S_{(g)}$.

We must verify that this assignment forms a sheaf. The key insight is that on the open set $D_+(f)$, the geometry is purely affine.

Theorem 11 Affine Covering.

For any homogeneous $f \in S_+$, there is a canonical homeomorphism of topological spaces:

$$\pi : D_+(f) \xrightarrow{\sim} \operatorname{Spec}\left(S_{(f)}\right).$$

Furthermore, under this identification, the restriction of our presheaf to $D_+(f)$ coincides with the structure sheaf of the affine scheme $\operatorname{Spec}\left(S_{(f)}\right)$.

Proof.

We define maps between the sets of prime ideals.

$$X = D_+(f) \subset \operatorname{Proj} S, \quad Y = \operatorname{Spec} S_{(f)}.$$

Step 1: The Map $\Phi : X \to Y$ (Contraction) Let $\mathfrak{p} \in D_+(f)$. This is a homogeneous prime of S not containing f. We define $\Phi(\mathfrak{p}) := \mathfrak{p}S_f \cap S_{(f)}$. This is a prime ideal in $S_{(f)}$ because it is the contraction of the prime ideal $\mathfrak{p}S_f \subset S_f$ under the inclusion $S_{(f)} \hookrightarrow S_f$.

Step 2: The Map $\Psi : Y \to X$ (Extension) Let $\mathfrak{q} \in \operatorname{Spec} S_{(f)}$. We construct a homogeneous ideal in S. First, consider the subset of homogeneous elements in S:

$$J(\mathfrak{q}) := \bigcup_{k \geq 0} \left\{ x \in S_k \,\middle|\, \frac{x^d}{f^k} \in \mathfrak{q} \right\}.$$

Let $\Psi(\mathfrak{q})$ be the ideal generated by $J(\mathfrak{q})$ in S. (Note: Since $\deg(f) = d$, the term x^d/f^k has degree $kd - kd = 0$, so it lies in $S_{(f)}$).

Lemma 12 Primality of $\Psi(\mathfrak{q})$.

The ideal $\Psi(\mathfrak{q})$ is a homogeneous prime ideal not containing f.

Proof. Step 1: Ideal structure

It suffices to check that if $x, y \in S_k$ satisfy the condition, then $x + y$ does. Note that in $S_{(f)}$, if $a, b \in \mathfrak{q}$, then $a + b \in \mathfrak{q}$. The condition $x^d/f^k \in \mathfrak{q}$ is equivalent to saying $x/1 \in \sqrt{\mathfrak{q}S_f}$ in the ring S_f. Since the radical of an ideal is an ideal, $x/1, y/1 \in \sqrt{\mathfrak{q}S_f} \implies (x + y)/1 \in \sqrt{\mathfrak{q}S_f}$. Thus $(x + y)^N/f^M \in \mathfrak{q}S_f$ for some large powers. Since $\mathfrak{q}$ is prime, this eventually implies $(x + y)^d/f^k \in \mathfrak{q}$. Thus $\Psi(\mathfrak{q})$ is a homogeneous ideal.

Step 2: Primality

Let x, y be homogeneous elements such that $xy \in \Psi(\mathfrak{q})$. Let $\deg x = n, \deg y = m$. By definition, this means:

$$\frac{(xy)^d}{f^{n+m}} \in \mathfrak{q}.$$

We can factor this element in $S_{(f)}$:

$$\frac{(xy)^d}{f^{n+m}} = \frac{x^d y^d}{f^n f^m} = \left(\frac{x^d}{f^n}\right) \cdot \left(\frac{y^d}{f^m}\right).$$

Since $\mathfrak{q}$ is a prime ideal in $S_{(f)}$, one of the factors must belong to $\mathfrak{q}$. If $\frac{x^d}{f^n} \in \mathfrak{q}$, then $x \in \Psi(\mathfrak{q})$. If $\frac{y^d}{f^m} \in \mathfrak{q}$, then $y \in \Psi(\mathfrak{q})$. Thus $\Psi(\mathfrak{q})$ is prime.

Step 3: Exclusion of f

$\frac{f^d}{f^d} = 1$. Since $\mathfrak{q}$ is proper, $1 \notin \mathfrak{q}$, so $f \notin \Psi(\mathfrak{q})$. □

Step 4: Bijectivity

i. $\Phi(\Psi(\mathfrak{q})) = \mathfrak{q}$: Let $z \in S_{(f)}$. Write $z = a/f^n$ with $\deg a = nd$. $z \in \Phi(\Psi(\mathfrak{q})) \iff a \in \Psi(\mathfrak{q}) \iff a^d/f^{nd} \in \mathfrak{q}$. Note that $a^d/f^{nd} = (a/f^n)^d = z^d$. Since $\mathfrak{q}$ is prime, $z^d \in \mathfrak{q} \iff z \in \mathfrak{q}$.

ii. $\Psi(\Phi(\mathfrak{p})) = \mathfrak{p}$: Let $x \in S_k$ be homogeneous. $x \in \Psi(\Phi(\mathfrak{p})) \iff x^d/f^k \in \mathfrak{p}S_f \cap S_{(f)} \iff x^d/f^k = p/f^m$ for some $p \in \mathfrak{p}$. This implies $f^m x^d \in \mathfrak{p}$ in S. Since $\mathfrak{p}$ is prime and $f \notin \mathfrak{p}$, this implies $x^d \in \mathfrak{p}$, and thus $x \in \mathfrak{p}$.

Thus Φ and Ψ are inverse bijections.

Step 4: The Homeomorphism

We show Φ is continuous and open. The closed sets of $\mathrm{Spec} S_{(f)}$ are of the form $V(\mathfrak{b})$ for ideals $\mathfrak{b} \subseteq S_{(f)}$. The closed sets of $D_+(f)$ are of the form $V_+(\mathfrak{a}) \cap D_+(f)$ for homogeneous ideals $\mathfrak{a} \subseteq S$.

Claim: $\Phi^{-1}(V(\mathfrak{b})) = V_+(\Psi(\mathfrak{b})) \cap D_+(f)$.

> *Proof.* Let $\mathfrak{p} \in D_+(f)$. $\Phi(\mathfrak{p}) \in V(\mathfrak{b}) \iff \mathfrak{b} \subseteq \mathfrak{p}S_f \cap S_{(f)}$. This is equivalent to saying that for every generator $b \in \mathfrak{b}$, $b \in \mathfrak{p}S_f$. Let $b = a/f^n$. $b \in \mathfrak{p}S_f \iff a \in \mathfrak{p}$. This is precisely the condition that $\mathfrak{p}$ contains the ideal in S generated by the numerators of $\mathfrak{b}$, which is exactly $\Psi(\mathfrak{b})$. Thus the topology matches. □

Step 5: The Isomorphism of Locally Ringed Spaces

We have established a homeomorphism $\pi : D_+(f) \to \operatorname{Spec} S_{(f)}$. We must show $\mathcal{O}_{\operatorname{Proj} S}|_{D_+(f)} \cong \pi^*\mathcal{O}_{\operatorname{Spec} S_{(f)}}$. It suffices to verify this on the basis of principal open sets. A basic open set in $\operatorname{Spec} S_{(f)}$ is $D(g)$ for $g \in S_{(f)}$. The preimage under π corresponds to the set of primes in $D_+(f)$ not containing the numerator of g. Let $g = h/f^n$ where $h \in S_{nd}$. The preimage is $D_+(h) \cap D_+(f) = D_+(fh)$. We check the rings of sections:

i. On the affine side: $\mathcal{O}_{\operatorname{Spec} S_{(f)}}(D(g)) = (S_{(f)})_g$.

ii. On the projective side: $\mathcal{O}_{\operatorname{Proj} S}(D_+(fh)) = S_{(fh)}$.

Algebraic Verification: $(S_{(f)})_g$ consists of elements $\frac{a/f^k}{(h/f^n)^m} = \frac{af^{nm}}{f^k h^m}$. Since these are degree 0 ratios, they are elements of degree 0 in the localization S_{fh}. Thus, there is a canonical isomorphism $(S_{(f)})_g \cong S_{(fh)}$. This isomorphism is compatible with restrictions. Finally, since we have an isomorphism of sheaves on a basis covering the space, we have an isomorphism of locally ringed spaces.

□

Corollary 13 Projective Schemes are Schemes.

The locally ringed space $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ is a scheme. For every $f \in S_+$, the open set $D_+(f)$ is an affine open subscheme isomorphic to $\operatorname{Spec}\left(S_{(f)}\right)$.

Example 14 Projective Space.

Let $S = k[x_0, \ldots, x_n]$ with the standard grading. Then $\operatorname{Proj} S = \mathbb{P}^n_k$. The open sets $D_+(x_i)$ cover $\mathbb{P}^n_k$. The ring of sections is $S_{(x_i)} = k[x_0/x_i, \ldots, x_n/x_i]$, which is a polynomial ring in n variables. Thus $\mathbb{P}^n_k$ is covered by $n+1$ copies of affine space $\mathbb{A}^n_k$.

We now turn to projective schemes. Just as affine schemes are built from rings and modules, projective schemes are built from graded rings and graded modules.

We will adopt the same philosophy as in the affine case: defining the sheaf directly on the basis of principal open sets using localization.

Let $S = \bigoplus_{d\geq 0} S_d$ be a graded ring. Let $S_+ = \bigoplus_{d>0} S_d$ be the irrelevant ideal. Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded S-module.

Definition 15 Degree-Zero Localization of Modules.

For a homogeneous element $f \in S$ of degree $d > 0$, the localization M_f inherits a natural $\mathbb{Z}$-grading (where $\deg(m/f^k) = \deg m - kd$). We define the **degree 0 part** of the localization as:

$$M_{(f)} := (M_f)_0 = \left\{ \frac{m}{f^k} \in M_f \;\middle|\; \deg(m) = k \cdot \deg(f) \right\}.$$

This $M_{(f)}$ is a module over the ring $S_{(f)} := (S_f)_0$.

Let $X = \operatorname{Proj} S$. The topology of X is generated by the basis of principal open sets:

$$D_+(f) := \{\mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p}\}$$

where $f \in S_+$ is homogeneous.

Definition 16 Projective Module Sheaf.

Let M be a graded S-module. We define a presheaf $\widetilde{M}$ on the basis $\{D_+(f)\}_{f \in S_+}$ by assigning the degree-0 localization:

$$\widetilde{M}(D_+(f)) := M_{(f)}.$$

For an inclusion $D_+(g) \subseteq D_+(f)$, which implies $g^n = fh$ for some homogeneous h, there is a natural localization map $M_{(f)} \to M_{(g)}$ (induced by further inverting g).

Does the assignment above define a sheaf?

In the affine case, we had to perform a tedious verification using the fact that $\mathsf{AffSch} \simeq \mathsf{CommRing}^{op}$. Here, we can leverage our previous hard work.

Proposition 17 Sheaf Property.

The presheaf $\widetilde{M}$ defined on the basis forms a sheaf.

Proof. We need to verify the sheaf axiom for an arbitrary basic covering. Let $D_+(f)$ be a basic open set, and let $D_+(f) = \bigcup_{i \in I} D_+(g_i)$ be a covering by basic open sets. Without loss of generality, we may assume $g_i \in S_{(f)}$ (by replacing g_i with $g_i^{\deg f}/f^{\deg g_i - 1}$ or similar standard homogenization tricks, or simply noting that $D_+(g_i) \cap D_+(f) = D_+(fg_i)$).

The key insight is the isomorphism to the affine case. We have a canonical homeomorphism $\psi : D_+(f) \xrightarrow{\sim} \operatorname{Spec}(S_{(f)})$. Under this map:

i. The basic open set $D_+(f)$ corresponds to the total space $\operatorname{Spec}(S_{(f)})$.

ii. The basic open subset $D_+(g_i) \subseteq D_+(f)$ corresponds to the principal open set $D(g_i/f^k) \subseteq \operatorname{Spec}(S_{(f)})$.

iii. The module $M_{(f)}$ is an $S_{(f)}$-module.

iv. The value $M_{(g_i)}$ is canonically isomorphic to the localization of the module $M_{(f)}$ at the element defining the open set.

Thus, the sequence required to be exact for $\widetilde{M}$ on the cover $\{D_+(g_i)\}$ of $D_+(f)$:

$$0 \longrightarrow M_{(f)} \longrightarrow \prod M_{(g_i)} \longrightarrow \prod M_{(g_i g_j)}$$

is **algebraically identical** to the sheaf sequence for the affine module sheaf $\widetilde{M_{(f)}}$ on the affine scheme $\mathrm{Spec}(S_{(f)})$.

We have proved that the affine construction satisfies the sheaf axioms for any principal covering. Therefore, $\widetilde{M}$ satisfies the sheaf axioms on the basis $\mathcal{B}$. By the Extension Lemma, it defines a unique sheaf on $\mathrm{Proj}\, S$. □

With the sheaf well-defined, we immediately recover the standard properties without further calculation.

Proposition 18 Properties of Projective Module Sheaves.

Let S be a graded ring and M a graded S-module.

1. **Stalks:** For every $\mathfrak{p} \in \mathrm{Proj}\, S$, the stalk is given by the degree 0 localization at the prime:
$$(\widetilde{M})_{\mathfrak{p}} \cong M_{(\mathfrak{p})}.$$
2. **Local Structure:** For any homogeneous $f \in S_+$, there is a canonical isomorphism of ringed spaces:
$$(D_+(f), \widetilde{M}|_{D_+(f)}) \cong (\mathrm{Spec}\, S_{(f)}, \widetilde{M_{(f)}}).$$
3. **Quasi-coherence:** $\widetilde{M}$ is a quasi-coherent $\mathcal{O}_X$-module. If S is Noetherian and M is finitely generated, then $\widetilde{M}$ is coherent.

Proof. (1) follows by taking the direct limit of sections over $D_+(f)$ containing $\mathfrak{p}$. Since $\varinjlim M_{(f)} = M_{(\mathfrak{p})}$, the stalk is as claimed.

(2) follows directly from our construction and the proof of the sheaf property.

(3) follows from (ii) because $\widetilde{M}$ is locally isomorphic to affine module sheaves, which is the definition of quasi-coherence.

□

Note 19 Loss of Information.

Unlike the affine case, the functor $M \mapsto \widetilde{M}$ is **not** an equivalence of categories. For example, the modules M and its truncation $M_{\geq d} = \bigoplus_{k \geq d} M_k$ define the same sheaf (they have the same localizations for large degrees). Information about the "irrelevant" ideal is lost. We will address this via the Serre twisting sheaves $\mathcal{O}(n)$.

We first address the finiteness condition using the geometry of affine charts rather than raw element chains.

Proposition 20 Noetherian Localization.

Let S be a Noetherian graded ring and M a finitely generated graded S-module. For any homogeneous $f \in S_+$, the degree-zero localization $M_{(f)}$ is a Noetherian $S_{(f)}$-module.

Proof. Consider the affine open set $D_+(f) \cong \operatorname{Spec}(S_{(f)})$. Since S is Noetherian, the localization S_f is Noetherian. The ring $S_{(f)}$ is the subring of degree 0 elements in S_f. Note that $S_f \cong S_{(f)}[T, T^{-1}]$ (where $\deg T = \deg f$). A subring of a Noetherian ring is not always Noetherian, but here $S_{(f)}$ is a direct summand, and explicitly, $S_{(f)} \cong (S/(f-1))_0$ implies $S_{(f)}$ inherits the Noetherian property.

Since M is finitely generated over S, the localization $M_{(f)}$ is finitely generated over $S_{(f)}$. A finitely generated module over a Noetherian ring is Noetherian. □

Instead of defining twisting sheaves via transition functions or manual gluing immediately, we introduce the **Shift Functor** on the category of graded modules.

Let M and N be graded S-modules. The tensor product $M \otimes_S N$ is naturally a graded S-module:

$$(M \otimes_S N)_n = \operatorname{span}_{\mathbb{Z}}\{m \otimes n \mid m \in M_d, n \in N_e, d + e = n\}/\sim$$

where the relation matches the S-action: $(sm) \otimes n = m \otimes (sn)$.

To handle sheaves on Proj$S$, we look at the basic open sets $D_+(f)$. The geometric sections over these sets correspond to the degree-zero localizations of the modules. We begin with a purely algebraic lemma.

Lemma 21 Tensor Product Localization.

Let S be a graded ring, and let M, N be graded S-modules. For any homogeneous element $f \in S$, there is a natural isomorphism of $S_{(f)}$-modules:

$$(M \otimes_S N)_{(f)} \cong M_{(f)} \otimes_{S_{(f)}} N_{(f)}.$$

Proof. Recall that $M_{(f)}$ denotes the sub-module of degree 0 elements in the localization M_f. Consider the map $\phi : M_{(f)} \times N_{(f)} \to (M \otimes_S N)_{(f)}$ defined by:

$$\left(\frac{m}{f^a}, \frac{n}{f^b}\right) \mapsto \frac{m \otimes n}{f^{a+b}}.$$

This map is well-defined and $S_{(f)}$-bilinear. By the universal property of the tensor product, it induces a homomorphism:

$$\Phi : M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to (M \otimes_S N)_{(f)}.$$

Conversely, any element in $(M \otimes_S N)_{(f)}$ is a sum of terms of the form $\frac{m \otimes n}{f^k}$ where $\deg(m) + \deg(n) = k \deg(f)$. We can rewrite this fractions as:

$$\frac{m \otimes n}{f^k} = \frac{m}{f^{\deg m/\deg f}} \otimes \frac{n}{f^{\deg n/\deg f}} \quad \text{(assuming simple degrees for intuition).}$$

Rigorously, since localization commutes with tensor products for non-graded rings (S_f-modules), and taking the degree 0 part is an exact functor compatible with this structure, Φ is an isomorphism. □

We now apply the algebraic result to the geometry of $X = \operatorname{Proj} S$.

Proposition 22 Sheafification of Tensor Products.

Let M and N be graded S-modules. There is a canonical isomorphism of $\mathscr{O}_X$-modules:

$$\widetilde{M \otimes_S N} \cong \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N}.$$

Proof. It suffices to show that these two sheaves are isomorphic on the standard basis of open sets $\mathcal{B} = \{D_+(f) \mid f \in S_+\}$.

Let $f \in S_+$ be homogeneous. By the definition of the associated sheaf, the sections over $D_+(f)$ are given by the degree-zero localization:

$$\widetilde{M \otimes_S N}(D_+(f)) = (M \otimes_S N)_{(f)}.$$

On the other hand, the tensor product of sheaves is the sheafification of the presheaf tensor product. However, on affine schemes, the sections of the tensor product of quasi-coherent sheaves are the tensor product of the sections. Thus:

$$(\widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N})(D_+(f)) \cong \widetilde{M}(D_+(f)) \otimes_{\widetilde{S}(D_+(f))} \widetilde{N}(D_+(f)) = M_{(f)} \otimes_{S_{(f)}} N_{(f)}.$$

By Lemma 21, we have a natural isomorphism $(M \otimes_S N)_{(f)} \cong M_{(f)} \otimes_{S_{(f)}} N_{(f)}$ for every f. These local isomorphisms are compatible with restrictions (properties of localization), hence they glue to a global isomorphism of sheaves. □

The most comfortable way to handle twisting is to define it first on the module level.

Definition 23 Shifted Module.

For a graded module M and an integer n, we define the shifted module $M(n)$ by shifting the grading:

$$M(n)_k := M_{n+k}.$$

Now, we define the geometric twisting sheaf simply as the sheaf associated with the shifted ring itself.

Definition 24 Twisting Sheaf.

The twisting sheaf $\mathscr{O}_{\mathrm{Proj}\,S}(n)$ is defined as:

$$\mathscr{O}(n) := \widetilde{S(n)}.$$

We now use Proposition 22 to give a clean proof for the structure of twisted sheaves. Recall that we define the twisting sheaf as $\mathscr{O}(n) = \widetilde{S(n)}$.

Proposition 25 Twisting Sheaf Isomorphism.

For any graded S-module M and integer n, there is a natural isomorphism:

$$\widetilde{M(n)} \cong \widetilde{M} \otimes_{\mathscr{O}_X} \mathscr{O}(n).$$

Proof. By definition, $\mathcal{O}(n) = \widetilde{S(n)}$. Substituting this into the right-hand side:

$$\widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}(n) = \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{S(n)}.$$

We apply Proposition 22 to the modules M and $S(n)$:

$$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{S(n)} \cong \widetilde{M \otimes_S S(n)}.$$

Now we look at the graded module $M \otimes_S S(n)$. There is a canonical isomorphism of graded S-modules:

$$\Psi : M \otimes_S S(n) \longrightarrow M(n), \quad m \otimes s \mapsto s \cdot m.$$

Let us check the degrees. If $m \in M_d$ and $s \in S(n)_e = S_{n+e}$, then $m \otimes s$ has degree $d + e$. The image $s \cdot m$ is in $M_{d+n+e} = M(n)_{d+e}$. Thus, Ψ preserves degrees. Since $S(n)$ is free of rank 1 generated by $1 \in S(n)_{-n}$, this map is an isomorphism.

Applying the exact functor $\sim$ (sheafification) to the isomorphism Ψ, we get:

$$\widetilde{M \otimes_S S(n)} \cong \widetilde{M(n)}.$$

Combining the chain of isomorphisms completes the proof. □

Note 26 Twisting as a Derived Concept.

By combining the tensor product proposition and the definition above, the twisted sheaf $\mathscr{F}(n)$ becomes a natural derived concept rather than an arbitrary definition:

$$\mathscr{F}(n) := \mathscr{F} \otimes_{\mathcal{O}_{\mathrm{Proj}\, S}} \mathcal{O}(n) \cong \widetilde{M} \otimes_{\mathcal{O}_{\mathrm{Proj}\, S}} \widetilde{S(n)} \cong \widetilde{M \otimes_S S(n)} \cong \widetilde{M(n)}.$$

This proves that twisting the sheaf $\widetilde{M}$ is equivalent to shifting the underlying module M, providing a powerful computational tool that avoids working with restrictions to open sets explicitly.

Proposition 27 Invertibility of Twisting Sheaves.

For any integers n, m, we have $\mathcal{O}(n) \otimes_{\mathcal{O}_X} \mathcal{O}(m) \cong \mathcal{O}(n + m)$. Consequently, $\mathcal{O}(n)$ is an invertible sheaf.

Proof. We use Proposition 25. Let $M = S(n)$. Then:

$$\mathcal{O}(n) \otimes \mathcal{O}(m) = \widetilde{S(n)} \otimes \mathcal{O}(m) \overset{25}{\cong} \widetilde{S(n)(m)}.$$

Recall the definition of the shifted module: $(S(n))(m)_k = S(n)_{m+k} = S_{n+m+k} = S(n + m)_k$. Thus $S(n)(m) = S(n + m)$ as graded modules. Therefore:

$$\widetilde{S(n)(m)} \cong \widetilde{S(n + m)} = \mathcal{O}(n + m).$$

In particular, $\mathcal{O}(n) \otimes \mathcal{O}(-n) \cong \mathcal{O}(0) \cong \mathcal{O}_X$, which proves invertibility. □

We now address the proposition, which establishes the dictionary between graded ring homomorphisms and morphisms of schemes.

Theorem 28 Morphisms from Graded Ring Homomorphisms.

Let $\phi : S \to T$ be a homomorphism of graded rings (preserving degrees). Let $U \subseteq \mathrm{Proj}T$ be the open set defined by:

$$U = \{\mathfrak{q} \in \mathrm{Proj}T \mid S_+ \not\subseteq \phi^{-1}(\mathfrak{q})\}.$$

Then ϕ induces a canonical morphism of schemes $f : U \to \mathrm{Proj}S$.

Proof. We cover $\mathrm{Proj}S$ by affine open sets $D_+(h)$ for $h \in S_+$. The preimage of such a set under the induced map should be the locus in $\mathrm{Proj}T$ where $\phi(h)$ does not vanish.

Since ϕ is a homomorphism, it induces maps on the localizations:

$$\phi_h : S_{(h)} \to T_{(\phi(h))}.$$

This induces a morphism of affine schemes $\mathrm{Spec}(T_{(\phi(h))}) \to \mathrm{Spec}(S_{(h)})$. Note that $\mathrm{Spec}(T_{(\phi(h))}) = D_+(\phi(h)) \subseteq \mathrm{Proj}T$. The union of these open sets $D_+(\phi(h))$ for all $h \in S_+$ is exactly U. These local morphisms glue together to form the global morphism $f : U \to \mathrm{Proj}S$. □

Proposition 29 Pullback and Pushforward.

With the notation above:

1. **Pullback:** For any graded S-module M, there is an isomorphism of $\mathcal{O}_X|_U$-modules:

$$f^*(\widetilde{M}) \cong (\widetilde{T \otimes_S M})|_U.$$

2. **Pushforward:** For any graded T-module N, regarding N as an S-module via ϕ (denoted ${}_SN$), we have:

$$f_*(\widetilde{N}|_U) \cong \widetilde{({}_SN)}.$$

Proof. The statements are local, so we check them on the distinguished affine basis. Let $h \in S_+$. The map f restricts to $f|_{D_+(\phi(h))} : D_+(\phi(h)) \to D_+(h)$.

Pullback: On the affine chart $D_+(h)$, the sheaf $\widetilde{M}$ corresponds to the module $M_{(h)}$. The pullback of a sheaf on affine schemes corresponds to the tensor product of the underlying module with the target ring. Thus, on $D_+(\phi(h))$:

$$f^*(\widetilde{M})|_{D_+(\phi(h))} \cong \left(M_{(h)} \otimes_{S_{(h)}} T_{(\phi(h))}\right)^{\sim}.$$

By Lemma 21 (applied to T as an S-module), this module is isomorphic to $(M \otimes_S T)_{(\phi(h))}$. This is exactly the section of $(\widetilde{M \otimes_S T})$ over $D_+(\phi(h))$.

Pushforward: We verify the sections over $D_+(h)$. By definition of the pushforward sheaf:

$$f_*(\widetilde{N}|_U)(D_+(h)) = \widetilde{N}(f^{-1}(D_+(h))) = \widetilde{N}(D_+(\phi(h))).$$

The right hand side is the localization $N_{(\phi(h))}$.

Now consider the sheaf $\widetilde{({}_S N)}$ on Proj$S$. Its sections over $D_+(h)$ are $({}_S N)_{(h)}$.

The isomorphism boils down to the identity:

$$N_{(\phi(h))} = \left\{ \frac{x}{\phi(h)^k} \right\} \quad \text{vs} \quad ({}_S N)_{(h)} = \left\{ \frac{x}{h^k} \right\}.$$

Since the S-action on N is defined by $s \cdot x = \phi(s)x$, the denominator h^k in the S-module structure acts exactly as $\phi(h)^k$ in the T-module structure. Thus the modules are identical. □

Note 30 Pullback of Twisting Sheaves.

Applying the pullback formula to the twisting sheaf $M = S(n)$:

$$f^* \mathscr{O}_{\mathrm{Proj}S}(n) = f^* \widetilde{S(n)} \cong (\widetilde{T \otimes_S S(n)})|_U \cong \widetilde{T(n)}|_U = \mathscr{O}_{\mathrm{Proj}T}(n)|_U.$$

This recovers the essential fact that the pullback of a line bundle is the line bundle of the same degree (restricted to the definition locus).

Before establishing the isomorphism between a quasi-coherent sheaf and its associated graded module, we must establish the geometric counterpart to algebraic localization. The following lemma asserts that the twisting sheaf $\mathscr{O}(1)$ provides enough sections to "clear denominators" globally. This property characterizes $\mathscr{O}(1)$ as an *ample* line bundle.

Lemma 31 Extension and Vanishing via Twisting.

Let $X = \mathrm{Proj}S$ (where S is finitely generated by S_1 over S_0) and let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Let $f \in S_1$ be a homogeneous element and $X_f = D_+(f)$ the corresponding basic open set.

1. **Vanishing:** If a global section $s \in \Gamma(X, \mathscr{F})$ vanishes when restricted to X_f, then it is annihilated by a power of f globally:

$$\exists n > 0 \text{ such that } s \otimes f^{\otimes n} = 0 \text{ in } \Gamma(X, \mathscr{F}(n)).$$

2. **Extension:** For any local section $t \in \Gamma(X_f, \mathscr{F})$, there exists a "twisted extension" to the whole space:

$$\exists n > 0 \text{ such that } t \otimes f^{\otimes n} \text{ extends to a global section in } \Gamma(X, \mathscr{F}(n)).$$

Proof. The geometry of X_f is affine, specifically $X_f \cong \mathrm{Spec}S_{(f)}$.

(i) Since X is quasi-compact, the vanishing of $s|_{X_f}$ implies s vanishes on a finite affine cover of X_f. Algebraically, if an element m in a module M maps to zero in the localization M_f, then $f^n m = 0$ for some n. Here, the section f plays the role of the localizing element.

(ii) Since $\mathscr{F}$ is quasi-coherent, its restriction to the affine open X_f corresponds to a module N. Elements of N (sections over X_f) are essentially fractions of the form "global section$/f^n$".

To extend t to X, we must "clear the denominator" by tensoring with $f^{\otimes n}$. This maps the section from $\mathscr{F}$ to $\mathscr{F}(n)$, allowing it to be defined globally. □

We now address the fundamental question: Can a quasi-coherent sheaf $\mathscr{F}$ be recovered from its graded module of sections?

Definition 32 Module of Twisted Global Sections.

For any $\mathcal{O}_X$-module $\mathscr{F}$, we define the associated graded S-module $\Gamma_*(\mathscr{F})$ by:

$$\Gamma_*(\mathscr{F}) := \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathscr{F}(n)).$$

Proposition 33 Canonical Isomorphism.

Let $\mathscr{F}$ be a quasi-coherent sheaf on $X = \operatorname{Proj} S$. There is a canonical isomorphism of $\mathcal{O}_X$-modules:

$$\Phi : \widetilde{\Gamma_*(\mathscr{F})} \xrightarrow{\sim} \mathscr{F}.$$

Proof. Let $M = \Gamma_*(\mathscr{F})$. We construct the morphism Φ and verify it is an isomorphism locally on the standard basis $\mathcal{B} = \{D_+(f) \mid f \in S_1\}$.

1. *Construction:* On the open set $D_+(f)$, the sheaf $\widetilde{M}$ corresponds to the degree-zero localization $M_{(f)}$. An element of $M_{(f)}$ is a fraction s/f^k where $s \in M_k = \Gamma(X, \mathscr{F}(k))$. We define the map:

$$\frac{s}{f^k} \longmapsto s|_{D_+(f)} \otimes (f|_{D_+(f)})^{-k}.$$

Since f is invertible on $D_+(f)$, this maps into $\mathscr{F}(k)|_{D_+(f)} \otimes \mathcal{O}(-k)|_{D_+(f)} \cong \mathscr{F}|_{D_+(f)}$.

2. *Surjectivity:* Let $t \in \mathscr{F}(D_+(f))$. By Lemma 31 (Extension), there exists $n > 0$ such that $t \otimes f^n$ extends to a global section $s \in \Gamma(X, \mathscr{F}(n)) = M_n$. In the localization $M_{(f)}$, we have the element s/f^n. Under Φ, this maps to $(t \otimes f^n) \otimes f^{-n} = t$. Thus, Φ is surjective.

3. *Injectivity:* Suppose $x \in \widetilde{M}(D_+(f))$ maps to zero. We can write $x = s/f^k$ for some global section $s \in M_k$. If $\Phi(x) = 0$, then $s|_{D_+(f)} = 0$ inside $\mathscr{F}(k)$. By Lemma 31 (Vanishing), there exists $m > 0$ such that $s \otimes f^m = 0$ in $\Gamma(X, \mathscr{F}(k+m))$. In the localization $M_{(f)}$, this implies:

$$\frac{s}{f^k} = \frac{s \cdot f^m}{f^{k+m}} = \frac{0}{f^{k+m}} = 0.$$

Thus, Φ is injective.

Since Φ is an isomorphism on every basic open set $D_+(f)$, it is a global isomorphism. □

Note 34 Adjunction Interpretation.

The construction $M \mapsto \widetilde{M}$ and $\mathscr{F} \mapsto \Gamma_*(\mathscr{F})$ form an adjoint pair:

$$\operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathscr{F}) \cong \operatorname{Hom}_{\mathrm{GrMod}_S}(M, \Gamma_*(\mathscr{F})).$$

Proposition 33 states that the *counit* of this adjunction is an isomorphism for quasi-coherent sheaves. This implies that QCoh(Proj$S$) is a reflective subcategory of graded modules (modulo torsion).

We previously established an adjunction between graded modules and sheaves. We now verify that for the standard polynomial ring, this adjunction is an equivalence. This confirms that our geometric definition of $\mathbb{P}^n$ matches our algebraic intuition of homogeneous polynomials.

Lemma 35 Global Sections of Projective Space.

Let A be a ring and $S = A[x_0, \dots, x_m]$ be the polynomial ring with the standard grading. Then the canonical homomorphism is an isomorphism:

$$S \xrightarrow{\sim} \Gamma_*(\mathscr{O}_{\mathrm{Proj} S}).$$

In other words, $\Gamma(\mathbb{P}^n_A, \mathscr{O}(k))$ is exactly the free A-module of homogeneous polynomials of degree k.

Proof. We determine the global sections by gluing sections on the standard cover $\mathcal{U} = \{D_+(x_i)\}_{i=0}^m$. The sheaf condition gives the exact sequence (part of the Čech complex):

$$0 \longrightarrow \Gamma(\mathrm{Proj} S, \mathscr{O}(k)) \longrightarrow \prod_{i=0}^{m} \Gamma(D_+(x_i), \mathscr{O}(k)) \longrightarrow \prod_{i,j} \Gamma(D_+(x_i) \cap D_+(x_j), \mathscr{O}(k)).$$

Identifying the geometric sections with localized rings:

$$\Gamma(D_+(x_i), \mathscr{O}(k)) \cong S(k)_{(x_i)} = A[x_0, \dots, x_m, x_i^{-1}]_k.$$

Thus, a global section is a collection of rational functions $\{\frac{P_i}{x_i^{d_i}}\}$ that agree on overlaps. Algebraically, this intersection is exactly the set of polynomials that have no poles at any $x_i = 0$. Since the ideal $(x_0, \dots, x_m)$ has codimension > 1 (it is the irrelevant ideal), a regular function defined outside this set extends uniquely to the whole space (by Hartogs's principle algebraic analogue). Thus, the intersection is exactly S_k. □

Using the Reconstruction Theorem and the calculation above, we can now translate perfectly between closed subschemes of projective space and homogeneous ideals.

Corollary 36 Projective Correspondence.

Let $\mathbb{P}^n_A = \mathrm{Proj} A[x_0, \dots, x_m]$.

1. **Closed Subschemes:** Let X be a closed subscheme of $\mathbb{P}^n_A$. Then $X \cong \mathrm{Proj}(S/I)$ for some homogeneous ideal $I \subset S$.
2. **Projective Schemes:** A scheme X is projective over Spec$A$ if and only if $X \cong \mathrm{Proj} S$ for some finitely generated graded algebra S (generated by S_1).

Proof. (i) A closed subscheme corresponds to a coherent ideal sheaf $\mathscr{I} \subset \mathscr{O}_{\mathbb{P}^n_A}$. By the

Reconstruction Theorem, $\mathscr{I} \cong \widetilde{\Gamma_*(\mathscr{I})}$. Let $I = \Gamma_*(\mathscr{I})$. Since $\mathscr{I}$ is a subsheaf of $\mathscr{O}_{\mathbb{P}^n_A}$, I is a submodule of $S \cong \Gamma_*(\mathscr{O}_{\mathbb{P}^n_A})$. Thus I is an ideal, and X is defined by I.

(ii) This follows immediately from (i) and the definition of a projective morphism (which factors through a closed immersion into $\mathbb{P}^n$). □

The power of projective geometry lies in the ability to understand arbitrary coherent sheaves via the twisting operation. The following theorem asserts that any coherent sheaf on a projective scheme is a quotient of a sum of line bundles.

Definition 37 Very Ample Sheaf.

Let X be a scheme over Spec$A$. An invertible sheaf $\mathcal{L}$ on X is **very ample** if there exists an immersion $i : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong i^*\mathscr{O}_{\mathbb{P}^n_A}(1)$.

To prove the main theorem, we need a technical lemma that allows us to move sheaves between spaces.

Lemma 38 Projection Formula.

Let $f : X \to Y$ be a morphism of ringed spaces, $\mathscr{F}$ an $\mathscr{O}_X$-module, and $\mathscr{G}$ a locally free $\mathscr{O}_Y$-module of finite rank. Then:

$$f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G}) \cong f_*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}.$$

Proof. The map is constructed canonically. Since $\mathscr{G}$ is locally free, tensoring with it is a local operation that commutes with pushforward (which is checking sections on open sets). Locally on Y, $\mathscr{G} \cong \mathscr{O}_Y^{\oplus n}$, reducing the statement to the identity $f_*(\mathscr{F}^{\oplus n}) \cong (f_*\mathscr{F})^{\oplus n}$. □

Proposition 39 Serre's Theorem A.

Let X be a scheme proper over Spec$A$, let $\mathcal{L} = \mathscr{O}_X(1)$ be a very ample invertible sheaf, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module.

Then there exists an integer N such that for all $n \geq N$, the twisted sheaf $\mathscr{F}(n) := \mathscr{F} \otimes \mathcal{L}^{\otimes n}$ is **generated by global sections**. Equivalently, there exists a surjection:

$$\mathscr{O}_X^{\oplus m} \twoheadrightarrow \mathscr{F}(n).$$

Proof. Since $\mathcal{L}$ is very ample, we have a closed immersion $i : X \hookrightarrow \mathbb{P}^n_A$ (since X is proper, the immersion must be closed). We use the immersion to transfer the problem to $\mathbb{P}^n_A$.

Consider the pushforward sheaf $i_*\mathscr{F}$. Since i is a closed immersion, i_* is exact and preserves coherence. Furthermore, by the Projection Formula (Lemma 38):

$$i_*(\mathscr{F}(n)) \cong i_*(\mathscr{F} \otimes i^*\mathscr{O}_{\mathbb{P}^n_A}(n)) \cong (i_*\mathscr{F}) \otimes \mathscr{O}_{\mathbb{P}^n_A}(n) \cong (i_*\mathscr{F})(n).$$

Thus, $\mathscr{F}(n)$ is globally generated on X if and only if $(i_*\mathscr{F})(n)$ is globally generated on $\mathbb{P}^n_A$. We reduce to the case $X = \mathbb{P}^n_A$.

On $\mathbb{P}^n_A$, we cover the space by $D_+(x_i)$. Restricted to $D_+(x_i)$, the coherent sheaf corresponds to a finitely generated module M_i over the polynomial ring. Let $\{s_{ij}\}$ be a finite set of generators for M_i. By the Extension Lemma (from previous sections), for n large enough, $x_i^n s_{ij}$ extends to a global section of $\mathscr{F}(n)$. Since there are finitely many charts and finitely many generators, we can choose a single large N such that for all $n \geq N$, the global sections generate the sheaf at every point. □

8.2 Cohomology of Projective Space

Intuition 40 Cohomology of Projective Space.

In this section, we begin our study of the sheaf cohomology of projective spaces. The cohomological properties of projective space constitute a classic result in algebraic geometry: they are neither as trivial as the affine case nor as complex as those of general schemes. Characterized by high degrees of symmetry and duality, this topic represents a pinnacle in introductory algebraic geometry, both for its aesthetic elegance and its foundational utility.

The first theorem concerns the cohomology of twisting sheaves on projective space, famously known as 'Serre's Calculation'. Although the proof may appear lengthy and cumbersome, the core task within the Čech cohomology process is essentially the manipulation of polynomials. It could aptly be described as 'a linear algebra exercise masquerading as algebraic geometry.

Theorem 41 Cohomology of $\mathbb{P}^n_A$.

Let A be a Noetherian ring, let $S = A[x_0, \dots, x_n]$ be the polynomial ring with standard grading, and let $X = \mathbb{P}^n_A = \operatorname{Proj} S$. We consider the cohomology of the twisting sheaves $\mathscr{O}_X(k)$. It is convenient to compute the direct sum of all twists simultaneously:

$$\Gamma_*(\mathscr{O}_X) := \bigoplus_{k\in\mathbb{Z}} H^0(X, \mathscr{O}_X(k)) \quad \text{and generally} \quad M^i := \bigoplus_{k\in\mathbb{Z}} H^i(X, \mathscr{O}_X(k)).$$

Then:

(a) The natural map $S \to \Gamma_*(\mathscr{O}_X)$ is an isomorphism of graded S-modules.

(b) For intermediate degrees $0 < i < n$, $H^i(X, \mathscr{O}_X(k)) = 0$ for all k.

(c) $H^n(X, \mathscr{O}_X(-n-1)) \cong A$, and generally $M^n \cong \bigoplus_k H^n(X, \mathscr{O}_X(k))$ is a free A-module generated by monomials with strictly negative exponents ($x_0^{-1} \dots x_n^{-1}$ etc.).

(d) **Serre Duality (Perfect Pairing):** The natural map

$$H^0(X, \mathscr{O}_X(k)) \times H^n(X, \mathscr{O}_X(-k-n-1)) \longrightarrow H^n(X, \mathscr{O}_X(-n-1)) \cong A$$

is a perfect pairing of finitely generated free A-modules.

Proof. **Step 0: The Strategy (Geometry to Algebra Translation)** Since $X = \mathbb{P}^n_A$ is a separated Noetherian scheme, and $\mathscr{O}_X(k)$ is quasi-coherent, we can compute its cohomology using the **Čech complex** associated to any affine open cover.

We choose the standard affine cover $\mathcal{U} = \{U_0, \dots, U_n\}$, where $U_i = D_+(x_i) = \operatorname{Spec}(S_{(x_i)})$. Instead of computing one sheaf at a time, we define the graded sheaf $\mathscr{F} = \bigoplus_{k\in\mathbb{Z}} \mathscr{O}_X(k)$.

Crucially, the restriction of this graded sheaf to an intersection $U_{i_0\dots i_p}$ corresponds exactly to the localization of the graded ring S:

$$\mathscr{F}(U_{i_0\dots i_p}) = \bigoplus_{k\in\mathbb{Z}} \Gamma(D_+(x_{i_0}\dots x_{i_p}), \mathscr{O}(k)) \cong S_{x_{i_0}\dots x_{i_p}}.$$

This isomorphism preserves the grading. Thus, the Čech complex $C^\bullet(\mathcal{U}, \mathscr{F})$ becomes a complex of **graded S-modules**:

$$C^\bullet : \quad 0 \longrightarrow \prod_i S_{x_i} \xrightarrow{d^0} \prod_{i<j} S_{x_i x_j} \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} S_{x_0\dots x_n} \longrightarrow 0.$$

From this point on, the proof is purely about analyzing monomials in localized polynomial rings.

Step 1: Calculating H^0 (Part a) $H^0(X, \mathscr{F})$ is the kernel of d^0. An element $\alpha \in \prod S_{x_i}$ is a tuple $(\alpha_0, \dots, \alpha_n)$ where $\alpha_i \in S_{x_i}$. The condition $d^0(\alpha) = 0$ means $\alpha_i = \alpha_j$ inside $S_{x_i x_j}$ for all i, j.

Algebraically, S is a Unique Factorization Domain (UFD). If a rational function f can be written with only x_i in the denominator, and also with only x_j in the denominator, and this holds for all i, j, then f must be a polynomial (it has no denominators). Thus, $\operatorname{Ker}(d^0) = S$. Matching the grading, $H^0(X, \mathscr{O}(k)) = S_k$ (homogeneous polynomials of degree k).

Step 2: Calculating H^n (Part c) $H^n(X, \mathscr{F})$ is the cokernel of the last map d^{n-1}.

$$\bigoplus_{k=0}^{n} S_{x_0\dots\hat{x}_k\dots x_n} \xrightarrow{d^{n-1}} S_{x_0\dots x_n} \longrightarrow H^n \longrightarrow 0.$$

Let's analyze the monomials. $S_{x_0\dots x_n}$ is the free A-module generated by all monomials of the form $x_0^{l_0}\dots x_n^{l_n}$ where $l_i \in \mathbb{Z}$. The image of d^{n-1} comes from localizations where at least one variable, say x_k, is *not* inverted. This means the image consists of all Laurent monomials $x_0^{l_0}\dots x_n^{l_n}$ where **at least one exponent** $l_k \geq 0$.

Therefore, the quotient group (the cokernel) is isomorphic to the free A-module generated by the "leftover" monomials: those where **every exponent is strictly negative**.

$$H^n(X, \mathscr{F}) \cong \bigoplus_{l_0,\dots,l_n<0} A \cdot x_0^{l_0}\dots x_n^{l_n}.$$

Let's look at the grading (total degree $k = \sum l_i$). The "largest" monomial in H^n corresponds to $l_0 = \dots = l_n = -1$. The degree is $(-1) + \dots + (-1) = -n - 1$. Thus, $H^n(X, \mathscr{O}(k)) = 0$ for $k > -n - 1$. For $k = -n - 1$, the cohomology is generated by the single basis element $x_0^{-1}\dots x_n^{-1}$. Thus $H^n(X, \mathscr{O}(-n-1)) \cong A$.

Step 3: Serre Duality (Part d) We established:

- $H^0(X, \mathscr{O}(k))$ has basis $\{x_0^{a_0}\dots x_n^{a_n} \mid a_i \geq 0, \sum a_i = k\}$.
- $H^n(X, \mathscr{O}(-k-n-1))$ has basis $\{x_0^{b_0}\dots x_n^{b_n} \mid b_i < 0, \sum b_i = -k-n-1\}$.

The pairing into $H^n(X, \mathscr{O}(-n-1)) \cong A\cdot(x_0^{-1} \dots x_n^{-1})$ is given by multiplication of monomials.

$$(x^{\mathsf{a}}) \cdot (x^{\mathsf{b}}) = x^{\mathsf{a}+\mathsf{b}}.$$

Since $b_i < 0$ and $a_i \geq 0$, for the product to be $x_0^{-1} \dots x_n^{-1}$, we must have $a_i + b_i = -1$, or $b_i = -1 - a_i$. This correspondence $b_i \leftrightarrow -1 - a_i$ sets up a perfect bijection between the basis of H^0 and the basis of H^n. Since the bases are dual, the pairing is perfect (determinant is ± 1).

Step 4: Vanishing of Intermediate Cohomology (Part b) We prove $H^i(X, \mathscr{F}) = 0$ for $0 < i < n$ by induction on the dimension n.

Base Case ($n = 1$): There are no integers i such that $0 < i < 1$. The statement is vacuously true.

Inductive Step: Assume the theorem holds for $\mathbb{P}^{n-1}$. Consider x_n as an element of degree 1 in S. Multiplication by x_n gives an exact sequence of graded S-modules:

$$0 \longrightarrow S(-1) \xrightarrow{\cdot x_n} S \longrightarrow S/(x_n) \longrightarrow 0.$$

Geometrically, this corresponds to the sequence of sheaves on $X = \mathbb{P}^n$:

$$0 \longrightarrow \mathscr{F}(-1) \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}_H \longrightarrow 0,$$

where $H \cong \mathbb{P}^{n-1}$ is the hyperplane defined by $x_n = 0$. Consider the long exact sequence of cohomology. For any i, we have:

$$\dots \to H^i(X, \mathscr{F}(-1)) \xrightarrow{\cdot x_n} H^i(X, \mathscr{F}) \to H^i(H, \mathscr{F}_H) \to \dots$$

Note that $H^i(X, \mathscr{F}(-1))$ is just the graded module M^i shifted by 1. So the map is simply the multiplication map $M^i \xrightarrow{\cdot x_n} M^i$.

For $0 < i < n$:

- If $i < n - 1$: By induction, $H^i(H, \mathscr{F}_H) = 0$. The exact sequence becomes $M^i(-1) \xrightarrow{\cdot x_n} M^i$ is surjective.
- If $i = n - 1$ (and $n > 1$): We look at $H^{n-1}(H, \mathscr{F}_H) \to H^n(X, \mathscr{F}(-1))$. Wait, let's look at the algebra instead of the long sequence directly.

The Localization Argument (The Power Move): Let $M = H^i(X, \mathscr{F})$. Since this is computed via the Čech complex of S-modules, M is a graded S-module. What happens if we localize M at x_n? Localization is exact. $M_{x_n} = H^i(C^\bullet_{x_n})$. The complex $(C^\bullet)_{x_n}$ is the Čech complex for the cover of X intersected with $D_+(x_n)$. But $D_+(x_n)$ is affine! The cohomology of a quasi-coherent sheaf on an affine scheme vanishes for $i > 0$. Thus, $M_{x_n} = 0$.

Algebraically, this means for every element $m \in M$, there is some power k such that $x_n^k \cdot m = 0$.

Now combine this with the exact sequence from the hyperplane. For $0 < i < n - 1$, $H^i(\mathbb{P}^{n-1}) = 0$, so the map $x_n : M^i \to M^i$ is bijective. If an operator is bijective but locally nilpotent (annihilated by power), the module must be zero. (If $xm = 0$ and x is injective, $m = 0$). Thus $H^i = 0$ for $0 < i < n - 1$.

The boundary case $i = n - 1$ requires checking the map $H^{n-1}(\mathscr{F}_H) \to H^n(\mathscr{F}(-1))$. Using the explicit calculation of H^{n-1} (negative monomials in $x_0, \ldots, x_{n-1}$) and H^n (negative monomials in $x_0, \ldots, x_n$), one can check injectivity/exactness directly to show $H^{n-1}(\mathbb{P}^n) = 0$. □

We present the calculated results of this theorem in the table below. It is evident that non-trivial values occur only at the two 'diagonal' positions.

Table 8.1: Cohomology Groups $H^i(X, \mathscr{O}_X(n))$ for $X = \mathbb{P}^r_A$

Order i	$n \geq 0$	$-r < n < 0$	$n \leq -r-1$
$i = 0$	Rank $\binom{n+r}{r}$	0	0
$0 < i < r$	0	0	0
$i = r$	0	0	Rank $\binom{-n-1}{-n-r-1}$
Special Case: $H^r(X, \mathscr{O}_X(-r-1)) \cong A$			

The second theorem concerns the cohomology of coherent sheaves on projective spaces, known as 'Serre's Finiteness Theorem' and 'Serre's Vanishing Theorem' (to be distinguished from the affine case). Finiteness states that the cohomology of a coherent sheaf is finitely generated; Vanishing indicates that after a coherent sheaf is sufficiently twisted, its higher cohomology groups disappear.

Theorem 42 Serre's Finiteness and Vanishing.

Let A be a Noetherian ring, and let X be a projective scheme over A. Let $\mathscr{O}_X(1)$ be a very ample invertible sheaf on X. Let $\mathscr{F}$ be a **coherent** sheaf on X. Then:

(a) **Finiteness:** For each $i \geq 0$, the cohomology group $H^i(X, \mathscr{F})$ is a **finitely generated** A-module.

(b) **Vanishing:** There exists an integer n_0 (depending on $\mathscr{F}$) such that for all $n \geq n_0$ and all $i > 0$:

$$H^i(X, \mathscr{F}(n)) = 0.$$

Proof. **Step 1: Reduction to Projective Space $\mathbb{P}^r_A$.** Since X is projective over A and $\mathscr{O}_X(1)$ is very ample, there exists a closed immersion $\iota : X \hookrightarrow P = \mathbb{P}^r_A$ such that $\mathscr{O}_X(1) \cong \iota^*\mathscr{O}_P(1)$. Let $\mathscr{G} = \iota_*\mathscr{F}$ be the direct image sheaf on P.

- Since ι is a closed immersion (a finite morphism) and $\mathscr{F}$ is coherent, $\mathscr{G}$ is a **coherent** sheaf on P.
- By the property of cohomology on closed subspaces (**Lemma 2.10**), we have isomorphisms:
$$H^i(X, \mathscr{F}(n)) \cong H^i(P, \mathscr{G}(n)).$$

Thus, it suffices to prove the theorem for the coherent sheaf $\mathscr{G}$ on the ambient space $P = \mathbb{P}^r_A$. From now on, we assume $X = \mathbb{P}^r_A$.

Step 2: Verification for Line Bundles (The Base Case). We first check if the theorem holds for sheaves of the form $\mathscr{E} = \bigoplus_{j=1}^{m} \mathscr{O}_X(q_j)$, a finite direct sum of twisted line bundles. By **Theorem 5.1 (Serre's Calculation)**, we know the explicit structure of $H^i(\mathbb{P}^r, \mathscr{O}(q))$:

- For $i = 0$, it is the degree q part of the polynomial ring S, which is a finite free A-module.
- For $0 < i < r$, it is zero.
- For $i = r$, it is dual to the degree $-q - r - 1$ part of S, which is a finite free A-module.
- For $i > r$, it is zero.

(a) Finiteness: Since cohomology commutes with finite direct sums, $H^i(X, \mathscr{E})$ is a direct sum of finitely generated free A-modules, hence finitely generated. **(b) Vanishing:** For any fixed q_j, $H^i(X, \mathscr{O}(q_j + n)) = 0$ for $i > 0$ as long as the degree shift moves us out of the non-zero range of H^r (i.e., $q_j + n > -r - 1$). Since there are finitely many q_j, we can choose n_0 large enough such that this holds for all summands. Thus, the theorem holds for $\mathscr{E}$.

Step 3: Descending Induction on i. We proceed by descending induction on the cohomological degree i.

- **Base Case ($i > r$):** By Grothendieck's Vanishing Theorem (since $\dim \mathbb{P}^r = r$), $H^i(X, \mathscr{F}) = 0$ for all $i > r$. The zero module is finitely generated, and vanishing is trivial.

Inductive Step: Assume the statements (a) and (b) hold for degree $i + 1$. We prove them for degree i (where $i > 0$). Since $\mathscr{F}$ is coherent on $\mathbb{P}^r$, a fundamental property of coherent sheaves on projective space (Serre's Theorem A) states that $\mathscr{F}$ can be written as a quotient of a finite sum of line bundles. That is, there exists a surjective morphism $\mathscr{E} \to \mathscr{F} \to 0$, where $\mathscr{E} = \bigoplus \mathscr{O}(q_j)$. Let $\mathcal{R}$ be the kernel. Since the category of coherent sheaves is abelian, $\mathcal{R}$ is also **coherent**. We have a short exact sequence:

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow 0.$$

Proof of (a) Finiteness for $H^i(X, \mathscr{F})$: Consider the long exact sequence of cohomology:

$$\cdots \longrightarrow H^i(X, \mathscr{E}) \longrightarrow H^i(X, \mathscr{F}) \longrightarrow H^{i+1}(X, \mathcal{R}) \longrightarrow \ldots$$

1. $H^i(X, \mathscr{E})$ is a finitely generated A-module (by Step 2).
2. $H^{i+1}(X, \mathcal{R})$ is a finitely generated A-module (by the inductive hypothesis on degree $i + 1$, applied to the coherent sheaf $\mathcal{R}$).

The module $H^i(X, \mathscr{F})$ is "sandwiched" between these two. Specifically: Let $M = H^i(X, \mathscr{F})$. It is an A-module. The image of $H^i(X, \mathscr{E})$ in M is a quotient of a finitely generated module, hence finitely generated. The quotient $M/\mathrm{Im}(H^i(\mathscr{E}))$ is isomorphic to a submodule of $H^{i+1}(X, \mathcal{R})$. Since A is **Noetherian**, any submodule of a finitely generated module is finitely generated. Since M is an extension of a f.g. module by a f.g. module, M itself is finitely generated. This proves (a).

Proof of (b) Vanishing for $H^i(X, \mathscr{F}(n))$ ($i > 0$): Twist the short exact sequence by n (flatness of twisting):

$$0 \longrightarrow \mathcal{R}(n) \longrightarrow \mathscr{E}(n) \longrightarrow \mathscr{F}(n) \longrightarrow 0.$$

Consider the relevant segment of the cohomology sequence:

$$\cdots \longrightarrow H^i(X, \mathscr{E}(n)) \longrightarrow H^i(X, \mathscr{F}(n)) \longrightarrow H^{i+1}(X, \mathcal{R}(n)) \longrightarrow \dots$$

We need to show the middle term vanishes for $n \gg 0$.

1. By Step 2, $H^i(X, \mathscr{E}(n)) = 0$ for $n \geq n_1$ (since $i > 0$).
2. By the inductive hypothesis on degree $i + 1$, applied to $\mathcal{R}$, there exists n_2 such that $H^{i+1}(X, \mathcal{R}(n)) = 0$ for $n \geq n_2$.

Let $n_0 = \max(n_1, n_2)$. For all $n \geq n_0$, the outer terms in the sequence are zero:

$$0 \longrightarrow H^i(X, \mathscr{F}(n)) \longrightarrow 0.$$

Therefore, $H^i(X, \mathscr{F}(n)) = 0$. This proves (b).

Conclusion: By induction, the finiteness holds for all $i \geq 0$. The vanishing holds for all $i > 0$. □

The final key theorem provides a necessary and sufficient condition for a sheaf to be ample. Crucially, this criterion is applicable to any proper scheme, not being restricted solely to the projective case.

Proposition 43 Serre's Criterion for Ampleness.

Let A be a Noetherian ring, and let X be a **proper** scheme over $\operatorname{Spec} A$. Let $\mathcal{L}$ be an invertible sheaf on X. Then the following conditions are equivalent:

(i) $\mathcal{L}$ is **ample**.

(ii) For every **coherent** sheaf $\mathscr{F}$ on X, there exists an integer n_0 (depending on $\mathscr{F}$) such that for all $i > 0$ and all $n \geq n_0$:

$$H^i(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n}) = 0.$$

Proof. **Direction (i) ⟹ (ii): Reduction to Serre's Vanishing**

Suppose $\mathcal{L}$ is ample. By the definition of ampleness (on a proper scheme over a Noetherian base), some tensor power $\mathcal{L}^{\otimes m}$ is **very ample** over $\operatorname{Spec} A$ for some $m > 0$. This implies there is an immersion $X \hookrightarrow \mathbb{P}^N_A$ such that $\mathscr{O}(1)|_X \cong \mathcal{L}^{\otimes m}$. Since X is proper over A, this immersion is a **closed immersion**, so X is a projective scheme.

We need to show vanishing for $\mathcal{L}^{\otimes n}$ for *all* large n, not just multiples of m. We use a division algorithm argument. Let $\mathscr{F}$ be any coherent sheaf. For any integer $r \in \{0, 1, \dots, m-1\}$, consider the coherent sheaf:

$$\mathscr{F}_r := \mathscr{F} \otimes \mathcal{L}^{\otimes r}.$$

Since X is projective (via the embedding given by $\mathcal{L}^{\otimes m}$), we can apply **Serre's Vanishing Theorem (Theorem 5.2)** to the sheaf $\mathscr{F}_r$ with respect to the very ample sheaf $\mathscr{H} = \mathcal{L}^{\otimes m}$. This theorem guarantees the existence of an integer k_r such that for all $k \geq k_r$ and all $i > 0$:

$$H^i(X, \mathscr{F}_r \otimes \mathscr{H}^{\otimes k}) = 0.$$

Substituting the definitions, this means:

$$H^i(X, \mathscr{F} \otimes \mathcal{L}^{\otimes r} \otimes \mathcal{L}^{\otimes mk}) = H^i(X, \mathscr{F} \otimes \mathcal{L}^{\otimes (mk+r)}) = 0.$$

Let $n_0 = \max_{0 \leq r < m}\{mk_r + r\}$. Any integer $n \geq n_0$ can be written as $n = mk + r$ for some $r \in \{0, \ldots, m-1\}$ and $k \geq k_r$. Thus, $H^i(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$.

Direction (ii) $\Rightarrow$ (i): Establishing Global Generation

To prove $\mathcal{L}$ is ample, it suffices to show that for any coherent sheaf $\mathscr{F}$, the twisted sheaf $\mathscr{F} \otimes \mathcal{L}^{\otimes n}$ is **generated by global sections** for all $n \gg 0$.

Step 1: Pointwise Generation via H^1 Vanishing. Let $P \in X$ be a closed point. Let $\mathscr{I}_P$ be the ideal sheaf of P. We have the standard short exact sequence:

$$0 \longrightarrow \mathscr{I}_P \longrightarrow \mathscr{O}_X \longrightarrow k(P) \longrightarrow 0,$$

where $k(P)$ is the skyscraper sheaf supported at P. Tensor this sequence with $\mathscr{F} \otimes \mathcal{L}^{\otimes n}$ (which is right exact, and exact here since $\mathscr{F}$ is locally free or we analyze stalks... actually, tensoring preserves exactness of sheaf sequences if the rightmost term is a skyscraper or we just tensor the whole complex). More precisely, since $k(P)$ is a skyscraper, $\mathscr{F} \otimes \mathcal{L}^{\otimes n} \otimes k(P) \cong (\mathscr{F} \otimes \mathcal{L}^{\otimes n})|_P$. The exact sequence is:

$$0 \longrightarrow \mathscr{I}_P\mathscr{F} \otimes \mathcal{L}^{\otimes n} \longrightarrow \mathscr{F} \otimes \mathcal{L}^{\otimes n} \longrightarrow \mathscr{F} \otimes \mathcal{L}^{\otimes n} \otimes k(P) \longrightarrow 0.$$

Consider the long exact sequence of cohomology. The relevant part is:

$$\Gamma(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n}) \xrightarrow{\alpha} \Gamma(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n} \otimes k(P)) \longrightarrow H^1(X, \mathscr{I}_P\mathscr{F} \otimes \mathcal{L}^{\otimes n}).$$

By hypothesis (ii), applied to the coherent sheaf $\mathscr{I}_P\mathscr{F}$, there exists $n_0(P)$ such that for all $n \geq n_0(P)$, the H^1 term vanishes. Consequently, the map α is **surjective**.

Step 2: From Fiber to Neighborhood (Nakayama's Lemma). The term on the right, $\Gamma(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n} \otimes k(P))$, is simply the fiber of the sheaf at P, i.e., $(\mathscr{F} \otimes \mathcal{L}^{\otimes n})_P / \mathfrak{m}_P(\mathscr{F} \otimes \mathcal{L}^{\otimes n})_P$. The surjectivity of α means we can lift a basis of the fiber to global sections $s_1, \ldots, s_k \in \Gamma(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n})$. By **Nakayama's Lemma**, these sections generate the stalk $(\mathscr{F} \otimes \mathcal{L}^{\otimes n})_P$ as an $\mathscr{O}_{X,P}$-module. Since $\mathscr{F}$ is coherent (finite type), sections that generate the stalk at P will generate the sheaf over some open neighborhood U_P of P.

Step 3: Uniformity over X (The Arithmetic Trick). The above shows generation locally. We need global generation for **all** $n \geq N$. This requires a bit of care because $n_0(P)$ depends on P.

Take $\mathscr{F} = \mathscr{O}_X$. We proved that for any P, there is a neighborhood V and an integer $n_1 > 0$ such that $\mathcal{L}^{\otimes n_1}$ is generated by global sections on V. Applying the argument again for each $r \in \{0, 1, \dots, n_1 - 1\}$, there exists a neighborhood U_r of P and integer M_r such that $\mathscr{F} \otimes \mathcal{L}^{\otimes(M_r)}$ is generated by global sections. Wait, let's use the property of generated sheaves: if $\mathcal{L}^{\otimes n_1}$ is generated by sections, then its powers $(\mathcal{L}^{\otimes n_1})^{\otimes k}$ are also generated.

Let's refine the covering argument: For a fixed P, we found n_1 and V such that $\mathcal{L}^{\otimes n_1}$ is globally generated on V. For each $r \in \{0, \dots, n_1 - 1\}$, apply the local generation argument to $\mathscr{F} \otimes \mathcal{L}^{\otimes r}$. We find a neighborhood $V_r \subseteq V$ and a base threshold k_r such that $\mathscr{F} \otimes \mathcal{L}^{\otimes r} \otimes (\mathcal{L}^{\otimes n_1})^{\otimes k}$ is generated by global sections on V_r for $k \geq k_r$. (Note: If $\mathcal{A}$ and $\mathcal{B}$ are generated by global sections, so is $\mathcal{A} \otimes \mathcal{B}$. We are using sections of $\mathcal{L}^{n_1 k}$ and sections of $\mathscr{F} \otimes \mathcal{L}^r$).

Let $U_P = \bigcap_{r=0}^{n_1-1} V_r$. On this neighborhood, $\mathscr{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections for all $n \geq \max(n_1 k_r + r)$.

Since X is proper (hence quasi-compact), we can cover X by finitely many such neighborhoods $U_{P_1}, \dots, U_{P_j}$. Let N be the maximum of the thresholds for each neighborhood. Then for all $n \geq N$, $\mathscr{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections everywhere on X. Thus $\mathcal{L}$ is ample. □

8.3 Euler Characteristic and Arithmetic Genus

Intuition 44 Euler Characteristic and Arithmetic Genus.

In this section, we turn our attention to some more intriguing topics. The Euler characteristic is a fundamental concept in both homological algebra and differential manifolds; here, we define it in the context of sheaves. Given that a 'bundle' on a differential manifold is essentially a special case of a (locally free) sheaf, this definition serves as a natural generalization of the classical Euler characteristic. Furthermore, we will define the Hilbert polynomial and the arithmetic genus. While these concepts originated in classical algebraic geometry, we shall define them using modern language. As the reader will discover, these modern definitions remain perfectly compatible with the classical cases.

Definition 45 Euler Characteristic.

Let X be a projective scheme over a field k, and let $\mathscr{F}$ be a coherent sheaf on X. The **Euler characteristic** of $\mathscr{F}$ is defined as:

$$\chi(\mathscr{F}) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathscr{F}).$$

Note: By Serre's Finiteness Theorem, each $\dim_k H^i(X, \mathscr{F})$ is finite, and by Grothendieck's Vanishing Theorem, $H^i = 0$ for $i > \dim X$. Thus, this sum is well-defined and finite.

The reader will observe that our definition of the Euler characteristic for sheaves is essentially consistent with the conventional one. Moreover, it satisfies additivity, a fundamental result in homological algebra. It is the Serre Finiteness Theorem established earlier that ensures this concept is well-defined, by guaranteeing that the dimensions of all cohomology groups involved are finite.

Lemma 46 Additivity of Euler Characteristic.

The Euler characteristic is additive on short exact sequences. That is, if

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

is a short exact sequence of coherent sheaves on X, then:

$$\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'').$$

Proof. Consider the long exact sequence of cohomology derived from the short exact sequence:

$$0 \to H^0(\mathscr{F}') \to H^0(\mathscr{F}) \to H^0(\mathscr{F}'') \to H^1(\mathscr{F}') \to H^1(\mathscr{F}) \to \dots$$

Let $h^i(\cdot) = \dim_k H^i(X, \cdot)$. A standard result from linear algebra states that for any exact sequence of finite-dimensional vector spaces, the alternating sum of their dimensions is zero. Thus, taking the alternating sum over the entire long exact sequence:

$$\sum_{i \geq 0} (-1)^i \left(h^i(\mathscr{F}') - h^i(\mathscr{F}) + h^i(\mathscr{F}'') \right) = 0.$$

Rearranging the terms (which is valid since the sum is finite), we get:

$$\sum (-1)^i h^i(\mathscr{F}) = \sum (-1)^i h^i(\mathscr{F}') + \sum (-1)^i h^i(\mathscr{F}'').$$

By definition, this is $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$. □

We now introduce the definition of the Hilbert polynomial. Its central premise is that for a coherent sheaf on a projective scheme, the Euler characteristic of its twists, $\chi(X, \mathscr{F}(n))$, behaves as a polynomial function in n. The proof proceeds by induction on the dimension of the support. The most formidable challenge lies in defining a suitable map $\mathscr{F}(n-1) \to \mathscr{F}(n)$, [typically achieved by selecting a hyperplane section that does not contain the associated primes of the support].

Theorem 47 Existence of the Hilbert Polynomial.

Let X be a projective scheme over a field k, $\mathscr{O}_X(1)$ a very ample invertible sheaf, and $\mathscr{F}$ a coherent sheaf. Then there exists a polynomial $P(z) \in \mathbb{Q}[z]$ such that for all n sufficiently large:

$$\chi(\mathscr{F}(n)) = P(n).$$

This polynomial is called the **Hilbert Polynomial** of $\mathscr{F}$.

Proof. We proceed by induction on $d = \dim(\operatorname{Supp} \mathscr{F})$.

Base Case ($d = 0$): If the support of $\mathscr{F}$ has dimension 0, it consists of a finite set of closed points. In this case, $\mathscr{F}$ is a skyscraper sheaf (technically, an Artinian module over the local rings). Tensoring with $\mathscr{O}_X(n)$ does not change the dimension of the global sections because $\mathscr{O}_X(n)$ is locally trivial and the support is discrete (algebraically, $M \otimes_A A(n) \cong M$ if M has finite length and support at the maximal ideal?). More simply, $H^i(\mathscr{F}(n)) = 0$

for $i > 0$, and $H^0(\mathscr{F}(n)) \cong H^0(\mathscr{F})$ (non-canonically, dimension-wise). Thus, $\chi(\mathscr{F}(n))$ is a constant, which is a polynomial of degree 0.

Inductive Step ($d > 0$): Assume the theorem holds for all sheaves with support dimension $< d$. Let $X = \mathbb{P}^N_k$ (embedding via $\mathscr{O}(1)$). Let $x \in H^0(X, \mathscr{O}(1))$ be a linear form defining a hyperplane H. Consider the multiplication map $\mathscr{F}(-1) \xrightarrow{\cdot x} \mathscr{F}$. Let $\mathcal{K}$ be the kernel and $\mathcal{Q}$ be the cokernel. We have an exact sequence:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathscr{F}(-1) \xrightarrow{\cdot x} \mathscr{F} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Twisting this sequence by n (which is exact as $\mathscr{O}(n)$ is flat/locally free):

$$0 \longrightarrow \mathcal{K}(n) \longrightarrow \mathscr{F}(n-1) \longrightarrow \mathscr{F}(n) \longrightarrow \mathcal{Q}(n) \longrightarrow 0.$$

By the additivity of Euler characteristic (splitting the 4-term sequence into two short exact sequences):

$$\chi(\mathscr{F}(n)) - \chi(\mathscr{F}(n-1)) = \chi(\mathcal{Q}(n)) - \chi(\mathcal{K}(n)).$$

Note 48 Dimension Drops.

How do we ensure the dimension drops? The support of the cokernel $\mathcal{Q}$ is exactly $\operatorname{Supp}(\mathscr{F}) \cap H$. The support of the kernel $\mathcal{K}$ consists of points where x is a zero-divisor on $\mathscr{F}$.

To apply induction, we need $\dim \operatorname{Supp} \mathcal{K} < d$ and $\dim \operatorname{Supp} \mathcal{Q} < d$. Since $\mathscr{O}(1)$ is very ample, we can choose a hyperplane H (defined by x) such that H **does not contain any irreducible component** of $\operatorname{Supp}(\mathscr{F})$. (If the field k is infinite, this is easy by prime avoidance. If k is finite, we might strictly need to replace $\mathscr{O}(1)$ by a higher power, or extend scalars, but the polynomial result remains valid).

With such a choice:

1. x is not identically zero on any component of $\operatorname{Supp}(\mathscr{F})$, so $\dim(\operatorname{Supp}(\mathscr{F}) \cap H) < d$. Thus $\dim \operatorname{Supp} \mathcal{Q} < d$.

2. Since x is not a zero divisor on the generic points of $\mathscr{F}$, the kernel $\mathcal{K}$ is supported on a proper closed subset, so $\dim \operatorname{Supp} \mathcal{K} < d$.

By the induction hypothesis, $\chi(\mathcal{Q}(n))$ is a polynomial $P_Q(n)$ and $\chi(\mathcal{K}(n))$ is a polynomial $P_K(n)$ for large n. Let $R(n) = P_Q(n) - P_K(n)$. We have the relation:

$$\chi(\mathscr{F}(n)) - \chi(\mathscr{F}(n-1)) = R(n) \quad \text{for } n \gg 0.$$

A fundamental lemma in numerical polynomials states that if $f(n) - f(n-1)$ is a polynomial in n, then $f(n)$ is a polynomial in n (of degree one higher, roughly). Thus, $\chi(\mathscr{F}(n))$ is a polynomial for large n. $\square$

Finally, we introduce the concept of the arithmetic genus of a projective scheme, which is likewise defined in terms of the Euler characteristic. It is worth noting that when X is an integral scheme over an algebraically closed field (corresponding to a classical algebraic variety, even without the finite type assumption), the arithmetic genus can be expressed directly through the dimensions of the cohomology groups. In particular, when X is a curve (dimension 1), the arithmetic genus coincides with the dimension of the first cohomology group.

Definition 49 Arithmetic Genus.

Let X be a projective scheme of dimension r over a field k. The **arithmetic genus** $p_a(X)$ is defined by:

$$p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1).$$

Note: This depends only on the intrinsic structure of X, not on the projective embedding.

Proposition 50 Genus of Integral Schemes.

If X is an **integral** projective scheme over an **algebraically closed** field k, then $H^0(X, \mathcal{O}_X) \cong k$. Consequently:

$$p_a(X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}(X, \mathcal{O}_X).$$

In particular, if X is a curve ($r = 1$), then $p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$.

Proof. **1. Global Sections are Constants:** Since X is proper over k, $H^0(X, \mathcal{O}_X)$ is a finite-dimensional k-algebra. Since X is integral, $H^0(X, \mathcal{O}_X)$ is an integral domain. A finite-dimensional domain over a field is a field (multiplication is injective $\implies$ surjective). Thus $H^0(X, \mathcal{O}_X)$ is a finite field extension of k. Since k is algebraically closed, there are no non-trivial finite extensions. Thus $H^0(X, \mathcal{O}_X) \cong k$, and $\dim_k H^0(X, \mathcal{O}_X) = 1$.

2. The Formula: Substitute $\chi(\mathcal{O}_X) = \sum_{j=0}^{r} (-1)^j h^j(\mathcal{O}_X)$ into the definition:

$$p_a(X) = (-1)^r \left([h^0 - h^1 + \cdots + (-1)^r h^r] - 1 \right).$$

Using $h^0 = 1$, the first term cancels the -1:

$$p_a(X) = (-1)^r \sum_{j=1}^{r} (-1)^j h^j(\mathcal{O}_X) = \sum_{j=1}^{r} (-1)^{r+j} h^j(\mathcal{O}_X).$$

Let $i = r - j$. As j goes from 1 to r, i goes from $r - 1$ to 0. The exponent becomes $r + (r - i) = 2r - i \equiv -i \pmod 2$. Thus,

$$p_a(X) = \sum_{i=0}^{r-1} (-1)^i h^{r-i}(X, \mathcal{O}_X).$$

3. The Curve Case ($r = 1$): The sum runs from $i = 0$ to 0.

$$p_a(X) = (-1)^0 h^{1-0}(\mathcal{O}_X) = h^1(\mathcal{O}_X).$$

□

8.4 Examples

Example 51 Euler Characteristic and Genus of $\mathbb{P}^k$.

Let $X = \mathbb{P}^k_A$ over a Noetherian ring A. We compute the invariants for the twisted structure sheaves $\mathscr{O}_X(n)$.

1. Euler Characteristic $\chi(\mathscr{O}_X(n))$**:** By **Serre's Calculation (Theorem 5.1)**, the cohomology of $\mathbb{P}^k$ is concentrated in degrees 0 and k.

- For $n \geq 0$: $H^0(X, \mathscr{O}(n))$ is the free A-module of homogeneous polynomials of degree n in $k+1$ variables. Its rank is the combinatorial number $\binom{n+k}{k}$. All higher cohomology vanishes.

$$\chi(\mathscr{O}_X(n)) = h^0 - h^1 + \cdots = \binom{n+k}{k}.$$

- For $n < 0$: The formula still holds as a polynomial. Specifically, for $n \ll 0$, only H^k is non-zero. By Serre Duality, $H^k(\mathscr{O}(n)) \cong H^0(\mathscr{O}(-n-k-1))^\vee$. The rank is $\binom{(-n-k-1)+k}{k} = \binom{-n-1}{k}$. Combinatorially, $\binom{n+k}{k} = (-1)^k\binom{-n-1}{k}$ is a standard identity.

Thus, the Hilbert polynomial of $\mathbb{P}^k$ is:

$$P_{\mathbb{P}^k}(z) = \binom{z+k}{k} = \frac{(z+k)(z+k-1)\dots(z+1)}{k!}.$$

2. Arithmetic Genus: Using the definition $p_a(X) = (-1)^k(\chi(\mathscr{O}_X) - 1)$:

$$\chi(\mathscr{O}_X) = P_{\mathbb{P}^k}(0) = \binom{k}{k} = 1.$$

Therefore:

$$p_a(\mathbb{P}^k) = (-1)^k(1-1) = 0.$$

This confirms that projective spaces are "genus zero" objects.

Example 52 Degree and Genus of a Plane Curve.

Let X be a curve in $\mathbb{P}^2_k$ defined by a single homogeneous polynomial $f \in S = k[x_0, x_1, x_2]$ of degree d. We view X as the closed subscheme $\mathrm{Proj}(S/(f))$.

1. The Fundamental Exact Sequence: We start with the sequence of graded S-modules:

$$0 \longrightarrow S(-d) \xrightarrow{\cdot f} S \longrightarrow S/(f) \longrightarrow 0.$$

Why is this exact?

- The map $S \to S/(f)$ is the canonical projection, so it is surjective (cokernel).
- The map $S(-d) \xrightarrow{\cdot f} S$ is multiplication by f. Since $S = k[x_0, x_1, x_2]$ is an **integral domain** and $f \neq 0$, this multiplication map is **injective** (no zero divisors).

- The shift $(-d)$ ensures the map preserves degree: an element of degree n in $S(-d)$ (which is degree $n-d$ in S) maps to degree $(n-d)+d=n$.

Applying the sheafification functor (which is exact), we obtain the short exact sequence of sheaves on $\mathbb{P}^2$:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(-d) \longrightarrow \mathscr{O}_{\mathbb{P}^2} \longrightarrow \iota_*\mathscr{O}_X \longrightarrow 0,$$

where $\iota : X \hookrightarrow \mathbb{P}^2$ is the inclusion. We will denote $\iota_*\mathscr{O}_X$ simply as $\mathscr{O}_X$.

2. The Hilbert Polynomial: We twist the sequence by $\mathscr{O}(n)$ (preserving exactness):

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(n-d) \longrightarrow \mathscr{O}_{\mathbb{P}^2}(n) \longrightarrow \mathscr{O}_X(n) \longrightarrow 0.$$

Using the additivity of the Euler characteristic (**Lemma 5.3**):

$$\chi(\mathscr{O}_X(n)) = \chi(\mathscr{O}_{\mathbb{P}^2}(n)) - \chi(\mathscr{O}_{\mathbb{P}^2}(n-d)).$$

Substituting the formula from Example 1 (with $k=2$):

$$P_X(n) = \binom{n+2}{2} - \binom{(n-d)+2}{2} = \frac{(n+2)(n+1)}{2} - \frac{(n-d+2)(n-d+1)}{2}.$$

Expanding the terms:

$$\begin{aligned} P_X(n) &= \frac{1}{2}\left[(n^2+3n+2) - ((n-d)^2 + 3(n-d) + 2)\right] \\ &= \frac{1}{2}\left[n^2+3n+2-(n^2-2dn+d^2+3n-3d+2)\right] \\ &= \frac{1}{2}\left[2dn - d^2 + 3d\right] \\ &= d \cdot n + \frac{3d-d^2}{2}. \end{aligned}$$

Interpretation:

- **Degree:** The leading coefficient is d, which matches the degree of the curve (intersection number with a line).
- **Constant Term:** The constant term is $\frac{3d-d^2}{2}$.

3. The Genus Formula: Since X is a curve (dimension 1), its arithmetic genus is defined as:

$$p_a(X) = 1 - \chi(\mathscr{O}_X).$$

We calculate $\chi(\mathscr{O}_X) = P_X(0)$:

$$\chi(\mathscr{O}_X) = \frac{3d-d^2}{2}.$$

Therefore:

$$p_a(X) = 1 - \frac{3d-d^2}{2} = \frac{2-3d+d^2}{2} = \frac{(d-1)(d-2)}{2}.$$

This recovers the famous **Degree-Genus Formula** for plane curves. (e.g., line $d=1 \implies p_a=0$; cubic $d=3 \implies p_a=1$).

Let $X \subset \mathbb{P}^N$ be a hypersurface defined by a homogeneous polynomial of degree d. The structure sheaf fits into the short exact sequence:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^N}(-d) \longrightarrow \mathscr{O}_{\mathbb{P}^N} \longrightarrow \mathscr{O}_X \longrightarrow 0.$$

1. Hilbert Polynomial: Using the additivity of the Euler characteristic and the known formula for projective space $\chi(\mathscr{O}_{\mathbb{P}^N}(k)) = \binom{k+N}{N}$:

$$P_X(n) = \binom{n+N}{N} - \binom{n-d+N}{N}.$$

2. Arithmetic Genus: The arithmetic genus is defined as $p_a(X) = (-1)^{\dim X}(\chi(\mathscr{O}_X) - 1)$. Note that $\dim X = N - 1$. Calculating $\chi(\mathscr{O}_X) = P_X(0)$:

$$\chi(\mathscr{O}_X) = \binom{N}{N} - \binom{N-d}{N} = 1 - \binom{N-d}{N}.$$

Using the combinatorial identity $\binom{-k}{r} = (-1)^r \binom{k+r-1}{r}$, we treat $\binom{N-d}{N}$ as $\binom{-(d-N)}{N}$:

$$\binom{N-d}{N} = (-1)^N \binom{(d-N)+N-1}{N} = (-1)^N \binom{d-1}{N}.$$

Substituting back into the genus formula:

$$\begin{aligned} p_a(X) &= (-1)^{N-1}\left([1 - (-1)^N \binom{d-1}{N}] - 1\right) \\ &= (-1)^{N-1}\left(-(-1)^N \binom{d-1}{N}\right) \\ &= (-1)^{2N-1}(-1)\binom{d-1}{N} \\ &= \binom{d-1}{N}. \end{aligned}$$

For example, a plane curve ($N = 2$) has $p_a = \binom{d-1}{2}$, consistent with the previous section.

Theorem 53 Invariants of Complete Intersections.

Let $X \subset \mathbb{P}^N$ be a **complete intersection** defined by m homogeneous polynomials $f_1, \dots, f_m$ of degrees $d_1, \dots, d_m$. The structure sheaf $\mathscr{O}_X$ is resolved by the **Koszul Complex** $\mathcal{K}_\bullet(f_1, \dots, f_m)$:

$$0 \to \mathcal{L}_m \to \cdots \to \mathcal{L}_2 \to \mathcal{L}_1 \to \mathscr{O}_{\mathbb{P}^N} \to \mathscr{O}_X \to 0.$$

The term $\mathcal{L}_k$ corresponds to the k-th exterior power of the direct sum of line bundles corresponding to the equations. Explicitly:

$$\mathcal{L}_k = \bigwedge^k \left(\bigoplus_{i=1}^m \mathscr{O}(-d_i) \right) \cong \bigoplus_{1 \le i_1 < \dots < i_k \le m} \mathscr{O}_{\mathbb{P}^N}\left(-\sum_{j=1}^k d_{i_j} \right).$$

1. Euler Characteristic via Inclusion-Exclusion: By the additivity of Euler characteristic on long exact sequences, $\chi(\mathscr{O}_X)$ is the alternating sum of the characteristics of the resolution terms:

$$\chi(\mathscr{O}_X(n)) = \sum_{k=0}^{m} (-1)^k \chi(\mathcal{L}_k(n)).$$

Substituting the binomial formula for each summand:

$$\chi(\mathscr{O}_X(n)) = \sum_{k=0}^{m} (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le m} \binom{n - (d_{i_1} + \cdots + d_{i_k}) + N}{N}.$$

(Note: The $k = 0$ term is $\binom{n+N}{N}$, corresponding to $\mathscr{O}_{\mathbb{P}^N}(n)$).

2. Arithmetic Genus: Setting $n = 0$ to find $\chi(\mathscr{O}_X)$, and using $p_a(X) = (-1)^{\dim X}(\chi(\mathscr{O}_X) - 1)$ where $\dim X = N - m$:

$$p_a(X) = (-1)^{N-m} \left[\left(\sum_{k=0}^{m} (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le m} \binom{N - \sum d_{i_j}}{N} \right) - 1 \right].$$

Proof. The proof consists of three main steps: constructing the algebraic Koszul complex, establishing its exactness via the Regular Sequence property, and computing the Euler characteristic via sheafification.

Step 1: Construction of the Algebraic Koszul Complex Let $S = k[x_0, \dots, x_N]$ be the graded polynomial ring. Let M be a graded S-module. For a single homogeneous element $f \in S$ of degree d, we define the elementary complex $K_\bullet(f)$ as:

$$0 \longrightarrow S(-d) \xrightarrow{\cdot f} S \longrightarrow 0,$$

concentrated in homological degrees 1 and 0. For the sequence of polynomials $\mathsf{f} = (f_1, \dots, f_m)$, the **Koszul complex** $K_\bullet(\mathsf{f})$ is defined as the tensor product of these elementary complexes over S:

$$K_\bullet(\mathsf{f}) := K_\bullet(f_1) \otimes_S K_\bullet(f_2) \otimes_S \cdots \otimes_S K_\bullet(f_m).$$

The terms of this complex are exterior powers of the free module $E = \bigoplus_{i=1}^{m} S(-d_i)$. Specifically, the term in degree k is:

$$K_k \cong \bigwedge^k E = \bigwedge^k \left(\bigoplus_{i=1}^{m} S(-d_i) \right) \cong \bigoplus_{1 \le i_1 < \cdots < i_k \le m} S\left(- \sum_{j=1}^{k} d_{i_j} \right).$$

The boundary maps are given by the standard differential of the exterior algebra contraction with the vector $(f_1, \dots, f_m)$.

Step 2: Exactness of the Complex (The Geometric Input) The homology of the Koszul complex $H_k(\mathsf{f})$ measures the relations between the generators.

- $H_0(\mathsf{f}) \cong S/(f_1, \dots, f_m)$.
- $H_k(\mathsf{f}) = 0$ for all $k > 0$ **if and only if** $f_1, \dots, f_m$ form a **regular sequence**.

Geometric justification: Since X is a **complete intersection** in $\mathbb{P}^N$, the codimension of X is exactly m. This implies that locally on the affine charts, the equations $f_1, \dots, f_m$ cut down the dimension by 1 at each step. Thus, $f_1, \dots, f_m$ form a regular sequence in the local rings $\mathscr{O}_{\mathbb{P}^N,x}$. While they might not be a regular sequence globally in S (due to the irrelevant ideal $(x_0, \dots, x_N)$), the failure of exactness is confined to modules of finite length (supported at the vertex of the affine cone). However, when we pass to sheaves, "finite length modules" become zero. Thus, the sheafified Koszul complex is **exact**.

Step 3: The Sheaf Resolution Applying the sheafification functor $\sim$ (which is exact) to the complex $K_\bullet(\mathrm{f})$, we obtain a locally free resolution of the structure sheaf $\mathscr{O}_X$:

$$0 \longrightarrow \mathcal{L}_m \longrightarrow \mathcal{L}_{m-1} \longrightarrow \cdots \longrightarrow \mathcal{L}_1 \longrightarrow \mathscr{O}_{\mathbb{P}^N} \longrightarrow \mathscr{O}_X \longrightarrow 0.$$

Here $\mathcal{L}_k = \widetilde{K_k}$. Using the explicit form from Step 1:

$$\mathcal{L}_k = \bigoplus_{|I|=k} \mathscr{O}_{\mathbb{P}^N}(-d_I), \quad \text{where } d_I = \sum_{i \in I} d_i.$$

Step 4: Calculation of Euler Characteristic We twist the entire resolution by $\mathscr{O}(n)$. The sequence remains exact. By the additivity of the Euler characteristic (**Lemma 5.3**):

$$\chi(\mathscr{O}_X(n)) = \sum_{k=0}^{m} (-1)^k \chi(\mathcal{L}_k(n)).$$

Note that the term $\mathscr{O}_{\mathbb{P}^N}(n)$ corresponds to $k = 0$ (empty sum of degrees is 0). Substituting the decomposition of $\mathcal{L}_k$:

$$\chi(\mathcal{L}_k(n)) = \sum_{|I|=k} \chi\left(\mathscr{O}_{\mathbb{P}^N}(n - d_I)\right).$$

We know for projective space that $\chi(\mathscr{O}_{\mathbb{P}^N}(d)) = \binom{d+N}{N}$. Therefore:

$$\chi(\mathscr{O}_X(n)) = \sum_{k=0}^{m} (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le m} \binom{n - (d_{i_1} + \cdots + d_{i_k}) + N}{N}.$$

This proves the formula for the Hilbert Polynomial.

Step 5: Calculation of Arithmetic Genus Recall the definition of arithmetic genus for a scheme of dimension r:

$$p_a(X) = (-1)^r (\chi(\mathscr{O}_X) - 1).$$

For a complete intersection defined by m equations in $\mathbb{P}^N$, the dimension is $r = N - m$. We calculate $\chi(\mathscr{O}_X)$ by setting $n = 0$ in the Hilbert polynomial derived in Step 4:

$$\chi(\mathscr{O}_X) = \sum_{k=0}^{m} (-1)^k \sum_{|I|=k} \binom{N - d_I}{N}.$$

Substituting this into the genus formula:

$$p_a(X) = (-1)^{N-m} \left[\left(\sum_{k=0}^{m} (-1)^k \sum_{|I|=k} \binom{N - d_I}{N} \right) - 1 \right].$$

This completes the proof. □

We apply the above machinery to some famous surfaces in algebraic geometry.

Example 54 The Segre Surface (Quadric in $\mathbb{P}^3$).

Consider the surface $Q \subset \mathbb{P}^3$ defined by the quadratic equation $xy - zw = 0$ (so $N = 3, d = 2$).

1. Invariants: Using the hypersurface formula:

$$p_a(Q) = \binom{2-1}{3} = \binom{1}{3} = 0.$$

A surface with arithmetic genus 0 is often rational.

2. Geometry ($\mathbb{P}^1 \times \mathbb{P}^1$): This surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Consider the **Segre embedding** $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ defined by:

$$([u_0 : u_1], [v_0 : v_1]) \longmapsto [u_0v_0 : u_0v_1 : u_1v_0 : u_1v_1].$$

Let the coordinates of $\mathbb{P}^3$ be $Z_{00}, Z_{01}, Z_{10}, Z_{11}$. Notice that $Z_{00}Z_{11} - Z_{01}Z_{10} = (u_0v_0)(u_1v_1) - (u_0v_1)(u_1v_0) = 0$. Thus the image lies in Q. One can construct an inverse map using coordinate ratios on the open sets where coordinates are non-zero (e.g., $u_0/u_1 = Z_{00}/Z_{10} = Z_{01}/Z_{11}$), proving isomorphism. This explains why Q contains two families of lines (rulings), corresponding to $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{p\}$.

Example 55 Fermat Cubic Surface.

Consider $X \subset \mathbb{P}^3$ defined by $x^3 + y^3 + z^3 + w^3 = 0$ ($N = 3, d = 3$).

1. Invariants:

$$p_a(X) = \binom{3-1}{3} = \binom{2}{3} = 0.$$

Thus, cubic surfaces are also rational surfaces (rationality is a deep result, not immediate from $p_a = 0$, but $p_a = 0$ is necessary).

2. The 27 Lines: A famous geometric fact is that every smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 lines (over an algebraically closed field). For the Fermat cubic, these can be written down explicitly (e.g., lines like $x = \lambda y, z = \mu w$ where $\lambda^3 = -1, \mu^3 = -1$).

Example 56 K3 Surface.

Consider a smooth hypersurface $X \subset \mathbb{P}^3$ of degree $d = 4$ (e.g., $x^4 + y^4 + z^4 + w^4 = 0$).

1. Invariants:

$$p_a(X) = \binom{4-1}{3} = \binom{3}{3} = 1.$$

Surfaces with trivial canonical bundle and $H^1(X, \mathcal{O}_X) = 0$ (which implies $p_a = 1$) are called **K3 surfaces** (named after Kummer, Kähler, and Kodaira). They occupy a central place in the classification of surfaces, analogous to elliptic curves in dimension 1.

Example 57 Veronese Surface in $\mathbb{P}^5$.

The Veronese surface $V \subset \mathbb{P}^5$ is the image of the 2-Veronese embedding $\nu_2 : \mathbb{P}^2 \longrightarrow \mathbb{P}^5$.

1. Determinantal Definition: If we label coordinates of $\mathbb{P}^5$ as z_{ij} corresponding to basis monomials $x_i x_j$, the condition that a point comes from $\mathbb{P}^2$ is that the matrix of coordinates is rank 1:

$$\operatorname{rank}\begin{pmatrix} z_{00} & z_{01} & z_{02} \\ z_{01} & z_{11} & z_{12} \\ z_{02} & z_{12} & z_{22} \end{pmatrix} \leq 1.$$

The vanishing of the 2×2 minors of this symmetric matrix gives the quadratic equations defining V.

2. Not a Complete Intersection: These minors generate the ideal, but there are too many of them for the codimension. V has dimension 2 in $\mathbb{P}^5$ (codimension 3), but requires 6 minors to define it. It is a standard example of a variety that is *not* a complete intersection.

3. Invariants: Since $V \cong \mathbb{P}^2$, its invariants are those of the projective plane:

$$p_a(V) = p_a(\mathbb{P}^2) = 0.$$

Its degree in $\mathbb{P}^5$ is $2^2 = 4$.

CHAPTER 9

Differential Geometry on Schemes

9.1 Relative Differentials and Tangent Spaces

We want to find a universal way to define "differential" and "tangent space".

Definition 1 Derivation.

Let $A \longrightarrow B$ be a homomorphism of rings. An A-derivation is a map $D : B \longrightarrow M$ into a B-module M that satisfies:

- $D(a) = 0$ for all $a \in A$
- $D(b_1 + b_2) = D(b_1) + D(b_2)$
- $D(b_1 b_2) = D(b_1)b_2 + b_1 D(b_2)$

for all $b_1, b_2 \in B$.

Definition 2 Module of Relative Differentials.

The module of relative differentials $\Omega_{B/A}$ is a B-module equipped with a unique A-derivation $d : B \longrightarrow \Omega_{B/A}$ such that for every B-module M and derivation $D : B \longrightarrow M$, there exists a unique B-module homomorphism $\phi : \Omega_{B/A} \longrightarrow M$ with $D = \phi \circ d$. In terms of category theory, the functor $\mathrm{Mod}_B \longrightarrow \mathrm{Sets}$ given by $M \mapsto \mathrm{Der}_A(B, M)$ is represented by $\Omega_{B/A}$:

$$\mathrm{Hom}_B(\Omega_{B/A}, M) \simeq \mathrm{Der}_A(B, M)$$

Proposition 3 Construction of Relative Differentials.

Let $\phi : B \otimes_A B \longrightarrow B$ be the multiplication map $b_1 \otimes_A b_2 \mapsto b_1 b_2$, and let $I = \ker \phi$. The pair $(I/I^2, d)$, where $d : B \longrightarrow I/I^2$ is defined by $d(b) = 1 \otimes_A b - b \otimes_A 1 + I^2$, gives the module of relative differentials $\Omega_{B/A}$.

Proof. First, we observe that I/I^2 carries a B-module structure. Consider the canonical isomorphism $B \otimes_A B/I \simeq B$. I/I^2 is naturally a $B \otimes_A B/I$-module via:

$$(b_1 \otimes_A b_2 + I) \cdot (i \otimes_A j + I^2) = (b_1 i) \otimes_A (b_2 j) + I^2$$

B acts on I/I^2 via left multiplication: $b \cdot (x + I^2) = (b \otimes_A 1)x + I^2$. This is equivalent to right multiplication because $(b \otimes_A 1 - 1 \otimes_A b) \in I$, so for any $x \in I$, $(b \otimes_A 1 - 1 \otimes_A b)x \in I^2$.

The map $d : B \to I/I^2$ given by $d(b) = 1 \otimes_A b - b \otimes_A 1 + I^2$ is an A-derivation. For any derivation $D : B \to M$, we define the B-linear map:

$$\psi : I/I^2 \to M, \quad i \otimes_A j + I^2 \mapsto iD(j)$$

and extend by linearity. Checking the composition:

$$\psi \circ d(b) = \psi(1 \otimes_A b - b \otimes_A 1 + I^2) = 1 \cdot D(b) - b \cdot D(1) = D(b)$$

Since $D(1) = 0$ for any derivation, we have $\psi \circ d = D$. The uniqueness of ψ follows from the fact that I is generated by elements of the form $1 \otimes b - b \otimes 1$. □

Proposition 4 Properties of Relative Differentials.

Let A be a ring and B, C be A-algebras. Let $D = B \otimes_A C$. Then:

1. $\Omega_{D/C} \simeq \Omega_{B/A} \otimes_B D$
2. For any multiplicative subset $S \subseteq B$, $\Omega_{S^{-1}B/A} \simeq S^{-1}\Omega_{B/A}$

Proof. (1) By Yoneda's Lemma, it suffices to show $\mathrm{Der}_C(D, M) \simeq \mathrm{Der}_A(B, M)$ for any D-module M. Define $\phi : \mathrm{Der}_C(D, M) \to \mathrm{Der}_A(B, M)$ by $d \mapsto d \circ T$, where $T : B \to D$ is $b \mapsto b \otimes_A 1$. Conversely, for $d' \in \mathrm{Der}_A(B, M)$, define $\psi(d') : D \to M$ by $\psi(d')(b \otimes_A c) = c \cdot d'(b)$. Calculation shows:

$$(\phi \circ \psi(d'))(b) = \psi(d')(b \otimes_1 1) = 1 \cdot d'(b) = d'(b)$$

$$(\psi \circ \phi(d))(b \otimes_A c) = c \cdot d(b \otimes_A 1) = d(1 \otimes_A c \cdot b \otimes_A 1) = d(b \otimes_A c)$$

where we used that d is a C-derivation, so $d(1 \otimes_A c) = 0$.

(2) For localization, define $\phi : \mathrm{Der}_A(S^{-1}B, M) \to \mathrm{Der}_A(B, M)$ via pre-composition with the localization map $L : B \to S^{-1}B$. For $d' \in \mathrm{Der}_A(B, M)$, define $\psi(d') : S^{-1}B \to M$ by the quotient rule:

$$\psi(d')\left(\frac{b}{s}\right) = \frac{d'(b)s - bd'(s)}{s^2}$$

Since any derivation d on $S^{-1}B$ must satisfy $d(1/s) = -d(s)/s^2$, the isomorphism follows from:

$$(\psi \circ \phi(d))\left(\frac{b}{s}\right) = \frac{d(b)s - d(s)b}{s^2} = d\left(\frac{b}{s}\right)$$

and $\phi \circ \psi(d')(b) = d'(b)$. This establishes $\Omega_{S^{-1}B/A} \simeq S^{-1}\Omega_{B/A}$. □

We denote the method of using the definition of the module of relative differentials and Yoneda's lemma as "DAYL". We now explore two fundamental exact sequences associated with these modules.

Proposition 5 First Exact Sequence.

Let $A \to B \to C$ be homomorphisms of rings. There is an exact sequence of C-modules:

$$\Omega_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega_{C/A} \xrightarrow{\beta} \Omega_{C/B} \to 0$$

Proof. Let M be any C-module. Define a map $\phi : \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M)$ by $D \mapsto D \circ \pi$, where $\pi : B \to C$ is the given homomorphism. The kernel is:

$$\ker \phi = \{D \in \mathrm{Der}_A(C, M) \mid D(\pi(b)) = 0, \forall b \in B\} = \mathrm{Der}_B(C, M)$$

The equality holds because if $D(\pi(b)) = 0$, then D is B-linear: $D(\pi(b) \cdot c) = \pi(b)D(c) + cD(\pi(b)) = \pi(b)D(c)$. This gives the exact sequence:

$$0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M)$$

Using the representability $\mathrm{Hom}_C(\Omega_{C/A}, M) \simeq \mathrm{Der}_A(C, M)$ and the adjunction $\mathrm{Hom}_B(\Omega_{B/A}, M) \simeq \mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, M)$, we obtain:

$$0 \to \mathrm{Hom}_C(\Omega_{C/B}, M) \to \mathrm{Hom}_C(\Omega_{C/A}, M) \to \mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, M)$$

Since this holds for all C-modules M, by "DAYL", we conclude the sequence of modules is exact:

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

□

Proposition 6 Second Exact Sequence.

Let I be an ideal of B, and let $A \to B \to C = B/I$ be ring homomorphisms. There is an exact sequence:

$$I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

Proof. By Proposition 5, we have $\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$. Since $C = B/I$, any B-derivation $D : C \to M$ must satisfy $D(b + I) = 0$ for all $b \in B$, thus $\Omega_{C/B} = 0$.

Consider the map $\phi : I \to \Omega_{B/A} \otimes_B C$ defined by $b \mapsto d(b) \otimes (1 + I)$. For $a, b \in I$:

$$\begin{aligned} \phi(ab) &= (d(b)a + d(a)b) \otimes (1 + I) \\ &= d(b) \otimes (a + I) + d(a) \otimes (b + I) \\ &= d(b) \otimes 0 + d(a) \otimes 0 = 0 \end{aligned}$$

Thus ϕ induces $\delta : I/I^2 \to \Omega_{B/A} \otimes_B C$. To show exactness, let M be a C-module. We define $g : \mathrm{Der}_A(B, M) \to \mathrm{Hom}_C(I/I^2, M)$ where $g(D)(i + I^2) = D(i)$. The kernel consists of derivations that vanish on I, which are precisely those that factor through $C = B/I$, i.e., $\mathrm{Der}_A(C, M)$. The result follows by "DAYL". □

Proposition 7 Cotangent Space at a Point.

Let k be a field and $(B, \mathfrak{n})$ be a local ring with residue field $B/\mathfrak{n} \simeq k$ induced by $k \to B$. Then the canonical morphism δ induces an isomorphism:

$$\mathfrak{n}/\mathfrak{n}^2 \simeq \Omega_{B/k} \otimes_B k$$

Proof. Applying Proposition 4 with $A = k$, $I = \mathfrak{n}$, and $C = k$, and noting that $\Omega_{k/k} = 0$, we have a surjection:

$$\mathfrak{n}/\mathfrak{n}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes_B k \to 0$$

To show injectivity, consider the dual map $\delta^* : \mathrm{Der}_k(B, k) \to \mathrm{Hom}_k(\mathfrak{n}/\mathfrak{n}^2, k)$, where $\delta^*(D)(n + \mathfrak{n}^2) = D(n)$. For any $f \in \mathrm{Hom}_k(\mathfrak{n}/\mathfrak{n}^2, k)$, we can define an $F \in \mathrm{Der}_k(B, k)$ by $F(b) = f(b - a + \mathfrak{n}^2)$, where $a \in k$ is the image of b in the residue field. This shows δ^* is surjective, hence δ is injective. Thus, δ is an isomorphism. □

Proposition 8 Differentials of Polynomial Rings.

Let $B = A[t_1, \ldots, t_n]$ be a polynomial ring over A. Then $\Omega_{B/A}$ is a free B-module of rank n with basis $\{dt_1, \ldots, dt_n\}$. Furthermore, if A is a K-algebra, the following exact sequence splits:

$$0 \to \Omega_{A/K} \otimes_A B \to \Omega_{B/K} \to \Omega_{B/A} \to 0$$

Proof. Consider the map $d : B \to B^n$ defined by $f \mapsto \left(\frac{\partial f}{\partial t_1}, \ldots, \frac{\partial f}{\partial t_n}\right)$. This map is an A-derivation (the "gradient" map). To check the universal property, let M be a B-module and $D : B \to M$ be any A-derivation. Let $x_i = D(t_i) \in M$. By the Leibniz rule, for any $f \in B$, we must have:

$$D(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial t_i} x_i$$

We define a B-module homomorphism $\phi : B^n \to M$ by $(f_1, \ldots, f_n) \mapsto \sum f_i x_i$. Clearly, $D = \phi \circ d$, and such ϕ is uniquely determined by the images of the basis vectors. This proves $\Omega_{B/A} \simeq B^n$, where dt_i corresponds to the i-th standard basis vector.

Now consider the case where A is a K-algebra. From the first exact sequence (Proposition 3), we have:

$$\Omega_{A/K} \otimes_A B \xrightarrow{\alpha} \Omega_{B/K} \to \Omega_{B/A} \to 0$$

To show the sequence is short exact and splits, it suffices to show that the dual map on derivations $g : \mathrm{Der}_K(B, M) \to \mathrm{Der}_K(A, M)$ (the restriction map) is surjective. Given any K-derivation $D_0 : A \to M$, we can extend it to $\bar{D} : B \to M$ by arbitrarily choosing $x_i \in M$ and setting $\bar{D}(t_i) = x_i$, then extending via the Leibniz rule:

$$\bar{D}(\sum a_\alpha t^\alpha) = \sum D_0(a_\alpha) t^\alpha + \sum a_\alpha \bar{D}(t^\alpha)$$

This extension provides a section, hence the sequence is split exact. □

Definition 9 Finite Type and Finite Presentation.

Let M be an A-module.

- M is of **finite type** if there is a surjection $A^k \to M \to 0$ for finite k.
- M is **finitely presented** if there is an exact sequence $A^m \to A^k \to M \to 0$ for finite m, k.

An A-algebra C is of **finite type** if $C \simeq A[t_1, \dots, t_n]/I$. It is **finitely presented** if I is also finitely generated.

Proposition 10 Differentials of Finite Type and Presentation.

Let C be an A-algebra of finite type (resp. finitely presented). Then $\Omega_{C/A}$ is a C-module of finite type (resp. finitely presented).

Proof. Let $C = B/I$ where $B = A[t_1, \dots, t_n]$. From the second exact sequence (Proposition 4):

$$I/I^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

We know $\Omega_{B/A}$ is a free B-module of rank n, so $\Omega_{B/A} \otimes_B C$ is a free C-module of rank n.

1. If C is of finite type, $\Omega_{C/A}$ is the quotient of a finitely generated module ($\Omega_{B/A} \otimes_B C$), thus $\Omega_{C/A}$ is of finite type.
2. If C is finitely presented, $I = (f_1, \dots, f_s)$ is finitely generated. Consequently, I/I^2 is a finitely generated C-module. Since both I/I^2 and $\Omega_{B/A} \otimes_B C$ are finitely generated, $\Omega_{C/A}$ is finitely presented.

□

Proposition 11 Differentials of Finite Separable Field Extensions.

Let L/K be a finite separable extension of fields. Then the module of relative differentials vanishes:

$$\Omega_{L/K} = 0$$

Proof. By the universal property of the module of relative differentials, it suffices to show that any K-derivation $D : L \to M$ into an L-module M must be the zero map.

Let $x \in L$ be any element, and let $f(T) \in K[T]$ be its minimal polynomial over K. Since D is a K-derivation, it acts on the relation $f(x) = 0$ as follows:

$$0 = D(f(x)) = f'(x)D(x)$$

Because the extension L/K is separable, the minimal polynomial $f(T)$ has no multiple roots in its splitting field, which implies that the derivative $f'(x) \neq 0$ in L. Since L is a field and $f'(x)$ is non-zero, it is invertible. Therefore, we must have:

$$D(x) = 0$$

Since this holds for all $x \in L$, we conclude that $\mathrm{Der}_K(L, M) = 0$ for any M, which implies $\Omega_{L/K} = 0$. □

To define the sheaf of relative differentials $\Omega_{X/Y}$ for a morphism of schemes $f : X \to Y$, we consider the diagonal morphism $\Delta : X \to X \times_Y X$. Since Δ is a local immersion, there exists an open subscheme $W \subseteq X \times_Y X$ containing $\Delta(X)$ such that $\Delta(X)$ is a closed subscheme of W. Let $\mathscr{I}$ be the ideal sheaf of $\Delta(X)$ in W.

Proposition 12 Sheaf of Relative Differentials.

The sheaf of relative differentials $\Omega_{X/Y}$ defined as $\Delta^*(\mathscr{I}/\mathscr{I}^2)$ is a quasi-coherent $\mathscr{O}_X$-module. In the affine case where $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, we have:

$$\Omega_{X/Y} \simeq \widetilde{\Omega_{B/A}}$$

Proof. On the affine level, the diagonal morphism $\Delta : \operatorname{Spec} B \to \operatorname{Spec}(B \otimes_A B)$ corresponds to the ring multiplication map $\phi : B \otimes_A B \to B$. The ideal $I = \ker \phi$ corresponds to the ideal sheaf $\mathscr{I}$. Thus, we have the following identification:

$$\Omega_{X/Y} = \Delta^*(\mathscr{I}/\mathscr{I}^2) = \Delta^*(\widetilde{I/I^2}) \simeq \widetilde{(I/I^2) \otimes_{B\otimes_A B} B}$$

Since the B-module structure on I/I^2 via $B \otimes_A B \to B$ is exactly the one we studied in Proposition 3, we have $(I/I^2) \otimes_{B\otimes_A B} B \simeq I/I^2$ as B-modules. Combined with the previously proven result $I/I^2 \simeq \Omega_{B/A}$, we obtain:

$$\Omega_{X/Y} \simeq \widetilde{\Omega_{B/A}}$$

This proves the assertion. For a general morphism $f : X \to Y$, let $W = \operatorname{Spec} B \subseteq X$ and $V = \operatorname{Spec} A \subseteq Y$ be affine open sets such that $f(W) \subseteq V$. Then $\Omega_{X/Y}|_W \simeq \widetilde{\Omega_{B/A}}$, which shows $\Omega_{X/Y}$ is quasi-coherent.

Furthermore, if f is locally of finite type, B is a finitely generated A-algebra. By Proposition 10, $\Omega_{B/A}$ is a finitely generated B-module, hence $\Omega_{X/Y}$ is a coherent $\mathscr{O}_X$-module. The same logic applies to the "locally of finite presentation" case. □

Proposition 13 Base Change for Differentials.

Let $f : X \to Y$ be a morphism of schemes, and $g : Z \to Y$ be any morphism. Let $W = X \times_Y Z$ be the fiber product, with $p : W \to X$ being the projection. Then:

$$\Omega_{W/Z} \simeq p^*\Omega_{X/Y}$$

Proof. Recall that a morphism of sheaves is an isomorphism if and only if it is an isomorphism on all stalks. For the sheaf $\widetilde{\Omega_{B/A}}$, the stalk at $\mathfrak{p} \in \operatorname{Spec} B$ is $(\Omega_{B/A})_{\mathfrak{p}}$. Since localization is an exact functor, the exactness of sequences and isomorphisms proven for modules (Proposition 2) carry over to the associated sheaves.

Consider the local affine case: $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, and $Z = \operatorname{Spec} C$. Then $W =$

$\operatorname{Spec}(B \otimes_A C)$. Let $D = B \otimes_A C$. The projection $p : W \to X$ is induced by the ring homomorphism $\phi : B \to D$ given by $b \mapsto b \otimes_A 1$. From Proposition 4, we have the module isomorphism:

$$\Omega_{D/C} \simeq \Omega_{B/A} \otimes_B D$$

By applying the sheafification functor $\sim$ to both sides, we have:

$$\Omega_{W/Z} \simeq \widetilde{\Omega_{D/C}} \simeq \widetilde{\Omega_{B/A} \otimes_B D} \simeq p^*(\widetilde{\Omega_{B/A}}) \simeq p^*\Omega_{X/Y}$$

Since this holds for any affine open cover, the global isomorphism $\Omega_{W/Z} \simeq p^*\Omega_{X/Y}$ is established. □

Proposition 14 The First Exact Sequence for Schemes.

Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. There is an exact sequence of sheaves of relative differentials on X:

$$f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

Proof. Consider the local affine case where $Z = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $X = \operatorname{Spec} C$. The morphisms correspond to ring homomorphisms $A \to B \to C$. From our previous results on modules (Proposition 5), we have the exact sequence:

$$\Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0$$

Applying the sheafification functor $\sim$, which is exact, and noting that $\widetilde{\Omega_{B/A} \otimes_B C} \simeq f^*\Omega_{Y/Z}$, we obtain the exact sequence of sheaves. Since exactness is a local property and can be checked on stalks, this globalizes to the required sequence of $\mathscr{O}_X$-modules. □

Proposition 15 The Second Exact Sequence for Schemes.

Let $f : X \to Y$ be a morphism of schemes, and let $i : Z \to X$ be a closed immersion with ideal sheaf $\mathscr{I}$. There is an exact sequence of sheaves on Z:

$$i^*(\mathscr{I}/\mathscr{I}^2) \longrightarrow i^*\Omega_{X/Y} \longrightarrow \Omega_{Z/Y} \longrightarrow 0$$

Proof. In the local affine setting, let $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$, and $Z = \operatorname{Spec}(A/I)$, so $\mathscr{I} = \widetilde{I}$. From Proposition 6, we established the exactness of:

$$I/I^2 \longrightarrow \Omega_{A/B} \otimes_A (A/I) \longrightarrow \Omega_{(A/I)/B} \longrightarrow 0$$

Sheafifying this sequence on Z, we identify $\widetilde{I/I^2}$ as the conormal sheaf restricted to Z, denoted by $i^*(\mathscr{I}/\mathscr{I}^2)$. The middle term corresponds to the pull-back $i^*\Omega_{X/Y}$. Since sheafification preserves exactness, the conclusion follows. □

Proposition 16 Differentials of Affine Space.

Let S be a scheme and $\mathbb{A}^n_S$ be the n-dimensional affine space over S. The sheaf of relative differentials is a free $\mathscr{O}_{\mathbb{A}^n_S}$-module of rank n:

$$\Omega_{\mathbb{A}^n_S/S} \simeq \mathscr{O}^n_{\mathbb{A}^n_S}$$

In particular, it is generated by the global sections $dt_1, \dots, dt_n$.

Proof. Let $V = \operatorname{Spec} A$ be an open affine subscheme of S. Then the inverse image of V in $\mathbb{A}^n_S$ is $U = \operatorname{Spec} A[t_1, \dots, t_n]$. On this affine open set, we have already shown in Proposition 8 that $\Omega_{A[t_1,\dots,t_n]/A}$ is a free module of rank n with basis $\{dt_i\}_{i=1}^n$. These local isomorphisms are canonical and glue together to form a global isomorphism $\Omega_{\mathbb{A}^n_S/S} \simeq \mathscr{O}^n_{\mathbb{A}^n_S}$. □

We work over $\mathbb{R}$ in the context of real smooth manifolds. Let $\mathscr{O}_p$ be the stalk of the sheaf of smooth functions at $p \in M$, with maximal ideal $\mathfrak{m} = \{f \in \mathscr{O}_p \mid f(p) = 0\}$. We define the following spaces:

- **Algebraic (AG):** $T_p^*(\text{AG}) = \mathfrak{m}/\mathfrak{m}^2$ and $T_p(\text{AG}) = (\mathfrak{m}/\mathfrak{m}^2)^*$.
- **Differential (DG):** $T_p(\text{DG}) = \operatorname{Der}_{\mathbb{R}}(\mathscr{O}_p, \mathbb{R})$ and $T_p^*(\text{DG}) = \mathscr{O}_p/\mathscr{G}_p$, where $\mathscr{G}_p$ consists of germs with vanishing first derivatives.
- **Kinematic (DM):** $T_p(\text{DM})$ as equivalence classes of curves $[\alpha]$.

Theorem 17 Equivalence of Tangent and Cotangent Spaces.

The algebraic, differential, and kinematic definitions of tangent and cotangent spaces at a point p on a smooth manifold M are all isomorphic.

Proof. **Step 1:** $T_p^*(\text{AG}) \simeq T_p^*(\text{DG})$.
Define $\phi : \mathfrak{m}/\mathfrak{m}^2 \to \mathscr{O}_p/\mathscr{G}_p$ by $f + \mathfrak{m}^2 \mapsto f + \mathscr{G}_p$ and $\psi : \mathscr{O}_p/\mathscr{G}_p \to \mathfrak{m}/\mathfrak{m}^2$ by $f + \mathscr{G}_p \mapsto (f - f(p)) + \mathfrak{m}^2$. To show ψ is well-defined, we use the Taylor expansion (Hadamard's Lemma) in a chart:

$$f(x) = f(0) + \sum_{i=1}^n x_i g_i(x), \quad \text{where } g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx)\,dt.$$

If $f \in \mathscr{G}_p$, then $\frac{\partial f}{\partial x_i}(0) = 0$, implying $g_i(0) = 0$, so $g_i \in \mathfrak{m}$. Thus $f - f(p) = \sum x_i g_i \in \mathfrak{m}^2$. This confirms the isomorphism of the cotangent spaces.

Step 2: $T_p(\text{AG}) \simeq T_p(\text{DG})$.
Define $\pi : (\mathfrak{m}/\mathfrak{m}^2)^* \to \operatorname{Der}_{\mathbb{R}}(\mathscr{O}_p, \mathbb{R})$ by $\pi(d)(f) = d((f - f(p)) + \mathfrak{m}^2)$. The inverse ρ sends a derivation D to the functional $\rho(D)(f + \mathfrak{m}^2) = D(f)$. Since D is a derivation into the residue field $\mathbb{R}$, $D(f \cdot g) = f(p)D(g) + g(p)D(f)$, and $D(\text{constant}) = 0$. One can check $\rho(D)$ vanishes on $\mathfrak{m}^2$ because $D(ab) = a(p)D(b) + b(p)D(a) = 0 \cdot D(b) + 0 \cdot D(a) = 0$ for $a, b \in \mathfrak{m}$.

Step 3: $T_p(\text{DM}) \simeq T_p(\text{DG})$.
Define $\tau : T_p(\text{DM}) \to T_p(\text{DG})$ by $\tau([\alpha])(f) = \frac{d(f\circ\alpha)}{dt}\Big|_{t=0}$. The Leibniz rule for $\tau([\alpha])$ follows

from the standard chain rule:

$$\tau([\alpha])(f\cdot g)=\frac{d}{dt}(f\circ\alpha\cdot g\circ\alpha)\Big|_0=f(p)\frac{d(g\circ\alpha)}{dt}\Big|_0+g(p)\frac{d(f\circ\alpha)}{dt}\Big|_0.$$

The inverse map η constructs a curve $\alpha_D(t)=\mathsf{u}^{-1}(tD^1,\dots,tD^m)$ for a given derivation $D=\sum D^i\frac{\partial}{\partial u^i}$. These maps are mutually inverse, identifying the geometric velocity of a curve with the algebraic notion of a derivation. □

Note 18 Tangent vs. Cotangent Spaces.

This identification shows that the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ is the more "primitive" algebraic object, and the tangent space is its dual. This matches the construction of $\Omega_{X/Y}$ as a representable functor.

Theorem 19 The Euler Sequence on $\mathbb{P}^n$.

Let A be a commutative ring, and let $X=\mathbb{P}^n_A=\operatorname{Proj}S$, where $S=A[x_0,\dots,x_n]$. There exists a canonical exact sequence of sheaves of $\mathcal{O}_X$-modules:

$$0\to\Omega_{X/A}\to\mathcal{O}_X(-1)^{\oplus(n+1)}\xrightarrow{\Delta}\mathcal{O}_X\to 0.$$

Here, $\Omega_{X/A}$ is the sheaf of relative differentials, and the map Δ is locally defined by the Euler derivation, corresponding to the multiplication map $(x_0,\dots,x_n)$ in the graded module.

Proof. **Step 1: Construction via Graded Modules**

Consider the graded S-module $L=S(-1)^{\oplus(n+1)}$. We denote the canonical basis elements of this free module by $e_0,\dots,e_n$. Since L is twisted by -1, the element e_i is considered to have degree 1 (so that $1\cdot e_i\in L_1$).

Define a homomorphism of graded S-modules $\phi: L\to S$ by sending the basis element e_i to the coordinate $x_i\in S_1$. Explicitly, for an element $\sum f_ie_i\in L$, we have:

$$\phi\left(\sum_{i=0}^n f_ie_i\right)=\sum_{i=0}^n f_ix_i.$$

This map is surjective in all degrees $d\geq 1$ (since the ideal $(x_0,\dots,x_n)$ generates the irrelevant ideal). Let $M=\ker(\phi)$ be the kernel graded module. We thus have an exact sequence of graded S-modules:

$$0\to M\to S(-1)^{\oplus(n+1)}\xrightarrow{\phi}S\to 0\quad\text{(modulo torsion at the irrelevant ideal).}$$

Applying the sheafification functor $\widetilde{(\cdot)}$, which is exact, we obtain the exact sequence of sheaves on X:

$$0\to\widetilde{M}\to\mathcal{O}_X(-1)^{\oplus(n+1)}\xrightarrow{\Delta}\mathcal{O}_X\to 0.$$

It remains to prove that $\widetilde{M}\cong\Omega_{X/A}$.

Step 2: Local Trivialization and Isomorphism Definition

We restrict our attention to the standard affine open cover. Let $U_k = D_+(x_k) = \operatorname{Spec} A[x_0/x_k, \dots, x_n/x_k]$. On this patch, the coordinate ring is $B_k = A[y_0^{(k)}, \dots, y_n^{(k)}]$ where $y_j^{(k)} = x_j/x_k$ (note $y_k^{(k)} = 1$).

The sheaf of differentials $\Omega_{U_k/A}$ is a free B_k-module generated by $\{d(x_j/x_k) \mid j \neq k\}$.

Now, let us examine the sheaf $\widetilde{M}$ on U_k. Sections of $\mathscr{O}_X(-1)$ on U_k are generated by the rational function x_k^{-1} (which has degree -1). Thus, a general section of $\mathscr{O}_X(-1)^{\oplus(n+1)}$ on U_k can be written as:

$$s = \sum_{j=0}^{n} g_j \cdot (x_k^{-1} e_j), \quad \text{where } g_j \in \mathscr{O}_X(U_k).$$

The map Δ sends this section to:

$$\Delta(s) = \sum_{j=0}^{n} g_j \frac{x_j}{x_k}.$$

For s to be in the kernel $\widetilde{M}(U_k)$, we must have $\sum g_j(x_j/x_k) = 0$.

We define a map $\psi_k : \Omega_{U_k/A} \to \widetilde{M}|_{U_k}$ by mapping the basis differentials to specific "Koszul-type" relations in the kernel. Define:

$$\psi_k\left(d\left(\frac{x_j}{x_k}\right)\right) = x_k^{-2}(x_k e_j - x_j e_k) = (1)\cdot(x_k^{-1}e_j) - \left(\frac{x_j}{x_k}\right)\cdot(x_k^{-1}e_k).$$

First, verify this image lies in the kernel:

$$\Delta\left(\psi_k(d(x_j/x_k))\right) = 1 \cdot \frac{x_j}{x_k} - \frac{x_j}{x_k}\cdot\frac{x_k}{x_k} = 0.$$

Second, observe that the elements $x_k^{-1}e_j - (x_j/x_k)x_k^{-1}e_k$ for $j \neq k$ form a basis for the kernel of the map $(g_0, \dots, g_n) \mapsto \sum g_j(x_j/x_k)$. Thus ψ_k is an isomorphism of $\mathscr{O}_{U_k}$-modules.

Step 3: Verification of Gluing (The Cocycle Condition)

To prove ψ_k defines a global isomorphism $\Omega_{X/A} \cong \widetilde{M}$, we must show that these local maps commute with the restriction maps on intersections $U_i \cap U_j$.

On the overlap $U_i \cap U_j$, we compute the image of a differential form using both maps. Recall the quotient rule for differentials:

$$d\left(\frac{x_m}{x_i}\right) = d\left(\frac{x_m}{x_j}\cdot\frac{x_j}{x_i}\right) = \frac{x_j}{x_i}d\left(\frac{x_m}{x_j}\right) + \frac{x_m}{x_j}d\left(\frac{x_j}{x_i}\right).$$

We apply the map ψ_j (defined on U_j) to the right-hand side expansion:

$$\text{RHS Image} = \frac{x_j}{x_i}\psi_j\left(d\left(\frac{x_m}{x_j}\right)\right) + \frac{x_m}{x_j}\psi_j\left(d\left(\frac{x_j}{x_i}\right)\right)$$

Let's compute $\psi_i(d(x_m/x_i))$:

$$A = \psi_i(d(x_m/x_i)) = \frac{1}{x_i^2}(x_i e_m - x_m e_i).$$

Now compute the image via ψ_j using the identity $d(u/v) = \frac{vdu-udv}{v^2}$:

$$d\left(\frac{x_m}{x_i}\right) = d\left(\frac{x_m/x_j}{x_i/x_j}\right) = \frac{(x_i/x_j)d(x_m/x_j) - (x_m/x_j)d(x_i/x_j)}{(x_i/x_j)^2}.$$

Applying ψ_j to this expression:

$$\begin{aligned}
\psi_j(\dots) &= \frac{(x_i/x_j)[x_j^{-2}(x_j e_m - x_m e_j)] - (x_m/x_j)[x_j^{-2}(x_j e_i - x_i e_j)]}{(x_i/x_j)^2} \\
&= \frac{x_j^2}{x_i^2}\left[\frac{x_i}{x_j^3}(x_j e_m - x_m e_j) - \frac{x_m}{x_j^3}(x_j e_i - x_i e_j)\right] \\
&= \frac{1}{x_i^2 x_j}\left[x_i(x_j e_m - x_m e_j) - x_m(x_j e_i - x_i e_j)\right] \\
&= \frac{1}{x_i^2 x_j}\left[x_i x_j e_m - x_i x_m e_j - x_m x_j e_i + x_m x_i e_j\right].
\end{aligned}$$

The terms $-x_i x_m e_j$ and $+x_m x_i e_j$ cancel out perfectly. We are left with:

$$= \frac{1}{x_i^2 x_j}(x_i x_j e_m - x_m x_j e_i) = \frac{x_j}{x_i^2 x_j}(x_i e_m - x_m e_i) = \frac{1}{x_i^2}(x_i e_m - x_m e_i).$$

This is exactly $\psi_i(d(x_m/x_i))$.

Since the local isomorphisms ψ_k define consistent maps on overlaps, they glue to a global isomorphism $\Omega_{X/A} \cong \widetilde{M}$. □

9.2 Nonsingular Variety and Geometric Genus

Definition 20 Differential Invariants of Schemes.

Let k be a field and X be a scheme over k. We assume X is *nonsingular* (or smooth) of dimension n, meaning that for every point $x \in X$, the local ring $\mathscr{O}_{X,x}$ is a regular local ring of dimension n. Under this hypothesis, the sheaf of Kähler differentials $\Omega_{X/k}$ is a locally free $\mathscr{O}_X$-module of rank n.

1. **Tangent Sheaf:** The tangent sheaf $\mathcal{T}_{X/k}$ is defined as the dual of the sheaf of differentials:

$$\mathcal{T}_{X/k} := \mathscr{H}om_{\mathscr{O}_X}(\Omega_{X/k}, \mathscr{O}_X).$$

2. **Exterior Powers:** For any $\mathscr{O}_X$-module $\mathscr{F}$ and integer $r \geq 0$, the exterior power sheaf $\Lambda^r\mathscr{F}$ is defined as the sheafification of the presheaf $U \mapsto \Lambda^r_{\mathscr{O}_X(U)}\mathscr{F}(U)$. Since $\Omega_{X/k}$ is locally free of rank n, the highest exterior power $\Lambda^n\Omega_{X/k}$ is a locally free sheaf of rank 1 (an invertible sheaf).

3. **Canonical Sheaf:** The canonical sheaf, denoted ω_X, is this top exterior power:

$$\omega_X := \Lambda^n \Omega_{X/k} = \det(\Omega_{X/k}).$$

4. **Geometric Genus:** If X is a *projective* scheme over k, the space of global sections $\Gamma(X, \omega_X)$ is a finite-dimensional vector space over k (by coherence and properness). The geometric genus $p_g(X)$ is defined as the dimension of this space:

$$p_g(X) := \dim_k \Gamma(X, \omega_X) = h^0(X, \omega_X).$$

5. **Birational Equivalence:** Two integral schemes X and Y are birationally equivalent if there exist dense open subschemes $U \subset X$ and $V \subset Y$ and an isomorphism of schemes $\phi : U \xrightarrow{\sim} V$.

Theorem 21 Birational Invariance of Geometric Genus.

Let X and Y be two nonsingular, projective varieties over a field k. If X and Y are birationally equivalent, then their geometric genera are equal:

$$p_g(X) = p_g(Y).$$

Proof. Since the relationship is symmetric, it suffices to show that $p_g(Y) \le p_g(X)$. Let $\phi : X \dashrightarrow Y$ be the rational map defining the birational equivalence. Let $V \subseteq X$ be the maximal open subset where ϕ is defined as a morphism, and let $f : V \to Y$ be this morphism. Since ϕ is a birational equivalence, there exists a dense open subset $U \subset V$ such that $f|_U : U \to f(U)$ is an isomorphism onto an open subset of Y.

Step 1: Pullback of Differentials. Consider the canonical morphism of sheaves on V induced by f:

$$f^*\Omega_{Y/k} \to \Omega_{V/k}.$$

Since f is an isomorphism on the dense open set U, this map is an isomorphism on U. Taking the n-th exterior power (where $n = \dim X = \dim Y$), we obtain a morphism of invertible sheaves on V:

$$\alpha : f^*\omega_Y \to \omega_V.$$

Restricted to U, $\alpha|_U$ is an isomorphism.

Step 2: Map of Global Sections. We define a map on global sections. A global section $s \in \Gamma(Y, \omega_Y)$ pulls back to a section $f^*s \in \Gamma(V, f^*\omega_Y)$. Composing with α, we get a section in $\Gamma(V, \omega_V)$. Since $\omega_V = \omega_X|_V$, we have a linear map:

$$\Psi : \Gamma(Y, \omega_Y) \to \Gamma(V, \omega_X|_V), \quad s \mapsto \alpha(f^*s).$$

Since ω_Y is locally free, any non-zero section s is non-zero on a dense open set. Its image via the isomorphism over U is non-zero, so Ψ is injective.

Step 3: Extension over Codimension ≥ 2 (The Critical Step). To conclude that $\dim \Gamma(Y, \omega_Y) \le \dim \Gamma(X, \omega_X)$, we must show that the image of Ψ lies in the subspace of sections that extend to all of X. That is, we need the restriction map $\rho : \Gamma(X, \omega_X) \to \Gamma(V, \omega_X|_V)$ to be an isomorphism.

The complement set is $Z = X \setminus V$. This is the locus where the rational map ϕ is not defined. Since Y is a *projective* (hence proper) scheme, we apply the **Valuative Criterion of Properness**. Let $x \in X$ be a point of codimension 1. Since X is nonsingular, the local ring $\mathcal{O}_{X,x}$ is a Discrete Valuation Ring (DVR). Let K be the function field of X. The birational map corresponds to a morphism $\operatorname{Spec} K \to Y$. By the valuative criterion, this lifts uniquely to a morphism $\operatorname{Spec} \mathcal{O}_{X,x} \to Y$. This implies that the rational map ϕ is defined at every point of codimension 1.

Therefore, the set of undefined points Z contains no points of codimension 1. Thus, $\operatorname{codim}(Z, X) \geq 2$.

Step 4: Algebraic Hartogs' Lemma. We invoke the property of normal schemes (nonsingular schemes are normal). Let $\mathcal{L}$ be a locally free sheaf (here $\mathcal{L} = \omega_X$) on a normal scheme X. If Z is a closed subset of codimension ≥ 2, then the restriction map

$$\Gamma(X, \mathcal{L}) \xrightarrow{\cong} \Gamma(X \setminus Z, \mathcal{L})$$

is a bijection.

Combining Step 2 and Step 4, we have an injection $\Gamma(Y, \omega_Y) \hookrightarrow \Gamma(V, \omega_X|_V) \cong \Gamma(X, \omega_X)$. Thus, $p_g(Y) \leq p_g(X)$. By symmetry, $p_g(Y) = p_g(X)$. ☐

Theorem 22 Regularity and Locally Free Differentials.

Let k be an algebraically closed field. Let X be an irreducible, separated scheme of finite type over k of dimension n. The sheaf of relative differentials $\Omega_{X/k}$ is a locally free sheaf of rank n if and only if X is a nonsingular variety (i.e., every local ring $\mathcal{O}_{X,x}$ is a regular local ring).

Proof. The problem is local on X. Let $x \in X$ be a closed point. Since X is an irreducible variety, the local ring $\mathcal{O}_{X,x}$ is a Noetherian local domain of dimension n with residue field $k(x) \cong k$. The hypothesis that X is nonsingular implies that $\mathcal{O}_{X,x}$ is a regular local ring. By the fundamental properties of Kähler differentials, there exists a canonical isomorphism between the fiber of the sheaf of differentials at x and the Zariski cotangent space:

$$\Omega_{X/k} \otimes_{\mathcal{O}_X} k(x) \cong \mathfrak{m}_x/\mathfrak{m}_x^2,$$

where $\mathfrak{m}_x$ denotes the maximal ideal of $\mathcal{O}_{X,x}$. Since $\mathcal{O}_{X,x}$ is regular, the dimension of the cotangent space coincides with the Krull dimension of the ring, hence $\dim_k(\mathfrak{m}_x/\mathfrak{m}_x^2) = n$. By Nakayama's Lemma, the minimal number of generators of the $\mathcal{O}_{X,x}$-module $(\Omega_{X/k})_x$ is equal to the dimension of its fiber, which is n.

($\Rightarrow$) Suppose $\Omega_{X/k}$ is locally free of rank n. Then the fiber dimension $\dim_k(\Omega_{X/k} \otimes k(x))$ is exactly n. By the isomorphism above, $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = n$, which implies A is regular.

($\Leftarrow$) Assume that X is a nonsingular variety. We aim to show that the sheaf of differentials $\Omega_{X/k}$ is locally free of rank n. Since local freeness is an open condition and X is a scheme of finite type over an algebraically closed field k, it suffices to verify the freeness of the stalk $(\Omega_{X/k})_x$ for every closed point $x \in X$.

Consider now the generic point η of X. The stalk $(\Omega_{X/k})_\eta$ is isomorphic to the module of differentials of the function field, $\Omega_{K(X)/k}$. Since k is algebraically closed (hence perfect) and X is an algebraic variety of dimension n, the function field $K(X)$ is a separably generated extension of k with transcendence degree n. Consequently, the rank of $\Omega_{X/k}$ is $\dim_{K(X)} \Omega_{K(X)/k} = n$.

We invoke the criterion for freeness over a local domain:

> If M is a finitely generated module over a Noetherian local domain A, and the minimal number of generators of M (given by $\dim_{A/\mathfrak{m}}(M \otimes_A A/\mathfrak{m})$) equals the rank of M (given by $\dim_{K(A)}(M \otimes_A K(A))$), then M is a free A-module.

In our case, for the module $M = (\Omega_{X/k})_x$ over $A = \mathscr{O}_{X,x}$, both values are equal to n. Therefore, $(\Omega_{X/k})_x$ is a free $\mathscr{O}_{X,x}$-module of rank n. Since x was an arbitrary closed point, we conclude that $\Omega_{X/k}$ is a locally free sheaf of rank n on X. □

Corollary 23 Generic Smoothness.

Let X be a variety over an algebraically closed field k. Then there exists a non-empty open dense subscheme $U \subseteq X$ such that U is nonsingular.

Proof. Let η be the generic point of X, and let $K = \mathscr{O}_{X,\eta}$ be the function field of X. Since X is a variety, K is a finitely generated field extension of k. Assuming k is perfect (which holds since k is algebraically closed), the extension K/k is separably generated. Consequently, the module of differentials $\Omega_{K/k}$ is a finite-dimensional vector space over K with $\dim_K \Omega_{K/k} = \operatorname{tr.deg}_k K = \dim X = n$.

The sheaf $\Omega_{X/k}$ is a coherent $\mathscr{O}_X$-module whose stalk at the generic point is $\Omega_{K/k}$. Let η be the generic point of X. We have established that the stalk $(\Omega_{X/k})_\eta \cong \Omega_{K/k}$ is a vector space of dimension n over the function field $K = \mathscr{O}_{X,\eta}$.

Since X is a variety, we may choose an affine open neighborhood $V = \operatorname{Spec}(A) \subseteq X$, where A is a finitely generated domain over k with fraction field K. Let M be the finitely generated A-module associated with the coherent sheaf $\Omega_{X/k}|_V$. The condition at the generic point translates to the isomorphism of K-vector spaces $M \otimes_A K \cong K^n$.

We invoke the algebraic principle that freeness at the generic point extends to a dense open set for finitely generated modules. Explicitly, let $e_1, \ldots, e_n$ be a basis of $M \otimes_A K$ over K. Since M is finitely generated, we may clear denominators to assume, without loss of generality, that the images of these basis elements lie in M. These elements define an A-module homomorphism $\phi : A^n \to M$.

Consider the kernel and cokernel of ϕ, denoted by $K = \ker(\phi)$ and $C = \operatorname{cok}(\phi)$, which fit into the exact sequence:

$$0 \longrightarrow K \longrightarrow A^n \xrightarrow{\phi} M \longrightarrow C \longrightarrow 0.$$

Since tensoring with K is flat (it corresponds to localization at the zero ideal), the sequence remains exact after tensoring with K. Because $\phi \otimes K$ is an isomorphism by construction, we have $K \otimes_A K = 0$ and $C \otimes_A K = 0$. Since M is finitely generated over a Noetherian ring A, both K and C are finitely generated A-modules. The property that their localization at the generic point is zero implies that their support is a proper closed subset of $\operatorname{Spec}(A)$. Thus, there exists a non-zero element $f \in A$ that annihilates both K and C (or more precisely, $K_f = 0$ and $C_f = 0$).

Localization at f yields an isomorphism $\phi_f : A_f^n \xrightarrow{\sim} M_f$. Consequently, the sheaf $\Omega_{X/k}$ restricted to the basic open set $U = D(f) \subseteq V$ corresponds to the free A_f-module M_f, which means $\Omega_{X/k}|_U$ is a locally free sheaf of rank n. Since U is non-empty and open in an irreducible variety, it is dense. Finally, applying Theorem 22 to the open subscheme U, we conclude that U is nonsingular. □

Proposition 24 Conormal Sequence and Smoothness of Subschemes.

Let X be a nonsingular variety over k, and let $Y \subset X$ be an irreducible closed subscheme defined by the ideal sheaf $\mathscr{I}$. The following conditions are equivalent:

1. Y is a nonsingular variety over k.
2. The sheaf $\Omega_{Y/k}$ is locally free, and the canonical sequence of sheaves (the second fundamental sequence)

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_{X/k} \otimes_{\mathscr{O}_X} \mathscr{O}_Y \to \Omega_{Y/k} \to 0$$

is exact (i.e., the map $\delta : \mathscr{I}/\mathscr{I}^2 \to \Omega_{X/k}|_Y$ is injective).

Proof. Let $n = \dim X$ and $q = \dim Y$. Since X is nonsingular, $\Omega_{X/k}$ is locally free of rank n.

Assume (1) holds. Since Y is nonsingular, $\Omega_{Y/k}$ is locally free of rank q. We consider the exact sequence of terms $\mathscr{I}/\mathscr{I}^2 \to \Omega_{X/k}|_Y \to \Omega_{Y/k} \to 0$. At any closed point $y \in Y$, tensoring with the residue field $k(y)$ yields the exact sequence of vector spaces involving the Zariski tangent spaces:

$$\mathscr{I}_y/\mathfrak{m}_y\mathscr{I}_y \to T_y^*X \to T_y^*Y \to 0.$$

By dimension counting, $\dim T_y^*X = n$ and $\dim T_y^*Y = q$. The kernel has dimension $n - q$, which equals the codimension of Y in X. Since X is regular, the ideal $\mathscr{I}$ is locally generated by a regular sequence of length $n - q$, implying $\mathscr{I}/\mathscr{I}^2$ is a locally free sheaf of rank $n - q$. The map of locally free sheaves $\mathscr{I}/\mathscr{I}^2 \to \ker(\Omega_{X/k}|_Y \to \Omega_{Y/k})$ is surjective and both have the same rank, hence it is an isomorphism. Thus the sequence is exact on the left.

Assume (2) holds. Since $\Omega_{Y/k}$ is locally free, its rank must be constant. Let this rank be r. From the exactness of the sequence and the local freeness of $\Omega_{X/k}$ (rank n) and $\mathscr{I}/\mathscr{I}^2$ (rank determined by local generators), one can deduce relations between the dimension of Y and the rank r. Specifically, using the Jacobian criterion applied to the generators of $\mathscr{I}$, local freeness and exactness imply the Jacobian matrix has maximal rank, which forces the singular locus to be empty. Thus Y is nonsingular. □

Definition 25 Conormal and Normal Sheaves.

Let X be a nonsingular variety over a field k, and let $Y \subset X$ be a closed subscheme defined by the sheaf of ideals $\mathscr{I}$. Assume that Y is itself a nonsingular variety.

The *conormal sheaf* of Y in X, denoted by $\mathcal{N}^\vee_{Y/X}$, is defined as the sheaf of $\mathcal{O}_Y$-modules:

$$\mathcal{N}^\vee_{Y/X} := \mathscr{I}/\mathscr{I}^2.$$

Since Y is nonsingular in X, this sheaf is a locally free $\mathcal{O}_Y$-module of rank $r = \operatorname{codim}(Y, X)$.

The *normal sheaf* of Y in X, denoted by $\mathcal{N}_{Y/X}$, is the dual of the conormal sheaf:

$$\mathcal{N}_{Y/X} := \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{I}/\mathscr{I}^2, \mathcal{O}_Y).$$

The geometric significance of these sheaves is encapsulated in the exact sequence of locally free sheaves on Y derived from the second fundamental sequence of differentials:

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \Omega_{Y/k} \to 0.$$

Dualizing this sequence yields the connection between tangent bundles:

$$0 \to \mathcal{T}_{Y/k} \to \mathcal{T}_{X/k}|_Y \to \mathcal{N}_{Y/X} \to 0.$$

Proposition 26 The Adjunction Formula.

Let Y be a nonsingular subvariety of codimension r in a nonsingular variety X over k. There is a natural isomorphism of invertible sheaves (canonical sheaves):

$$\omega_Y \cong \omega_X \otimes_{\mathcal{O}_X} \Lambda^r \mathcal{N}_{Y/X} \cong (\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \det(\mathcal{N}_{Y/X}).$$

In the specific case where Y is a divisor (codimension $r = 1$) defined by an invertible sheaf $\mathcal{L}$ (i.e., $\mathscr{I} \cong \mathcal{L}^{-1}$), the normal bundle is $\mathcal{N}_{Y/X} \cong \mathcal{L} \otimes \mathcal{O}_Y$. The formula simplifies to:

$$\omega_Y \cong \omega_X|_Y \otimes_{\mathcal{O}_Y} \mathcal{L}|_Y.$$

Proof. We begin with the short exact sequence of locally free $\mathcal{O}_Y$-modules:

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_{X/k}|_Y \to \Omega_{Y/k} \to 0.$$

For any short exact sequence of locally free sheaves $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$, there is a canonical isomorphism of their determinant bundles (top exterior powers): $\det(\mathcal{B}) \cong \det(\mathcal{A}) \otimes \det(\mathcal{C})$. Applying this to our sequence, noting that $\Omega_{X/k}|_Y$ has rank n and $\mathscr{I}/\mathscr{I}^2$ has rank r (where $n = \dim X$), we obtain:

$$\det(\Omega_{X/k}|_Y) \cong \det(\mathscr{I}/\mathscr{I}^2) \otimes \det(\Omega_{Y/k}).$$

The term on the left is $(\Lambda^n \Omega_{X/k}) \otimes \mathcal{O}_Y \cong \omega_X|_Y$. The term $\det(\Omega_{Y/k})$ is simply ω_Y. The term $\det(\mathscr{I}/\mathscr{I}^2)$ is $\Lambda^r \mathcal{N}^\vee_{Y/X}$. Rearranging the isomorphism by tensoring with the dual of the conormal determinant (which is the determinant of the normal bundle), we get:

$$\omega_Y \cong \omega_X|_Y \otimes \det(\mathcal{N}_{Y/X}).$$

For $r = 1$, $\mathcal{N}_{Y/X} \cong (\mathcal{L}^{-1})^\vee|_Y \cong \mathcal{L}|_Y$, yielding the divisor form of the adjunction formula. □

Example 27 Canonical Sheaf of $\mathbb{P}^n$ and Plane Curves.

1. The Canonical Sheaf of Projective Space. We determine $\omega_{\mathbb{P}^n}$ using the Euler exact sequence on $X = \mathbb{P}^n_k$:

$$0 \longrightarrow \Omega_{\mathbb{P}^n/k} \longrightarrow \mathscr{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \longrightarrow \mathscr{O}_{\mathbb{P}^n} \longrightarrow 0.$$

Taking the highest exterior powers (determinants) preserves the exactness relation in the Picard group. Thus:

$$\omega_{\mathbb{P}^n} \otimes \det(\mathscr{O}_{\mathbb{P}^n}) \cong \det(\mathscr{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}).$$

Since $\det(\mathscr{O}_{\mathbb{P}^n}) \cong \mathscr{O}_{\mathbb{P}^n}$, and the determinant of a direct sum is the tensor product of determinants, we have:

$$\omega_{\mathbb{P}^n} \cong \Lambda^{n+1}(\mathscr{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}) \cong \mathscr{O}_{\mathbb{P}^n}(-1)^{\otimes(n+1)} \cong \mathscr{O}_{\mathbb{P}^n}(-n-1).$$

2. Plane Curves and Geometric Genus. Let $C \subset \mathbb{P}^2$ be a nonsingular curve of degree d. C is a divisor defined by a section of $\mathcal{L} = \mathscr{O}_{\mathbb{P}^2}(d)$. Applying the adjunction formula with $X = \mathbb{P}^2$ and $\omega_X = \mathscr{O}_{\mathbb{P}^2}(-3)$:

$$\omega_C \cong \omega_{\mathbb{P}^2}|_C \otimes \mathscr{O}_{\mathbb{P}^2}(d)|_C \cong \mathscr{O}_{\mathbb{P}^2}(-3+d)|_C \cong \mathscr{O}_C(d-3).$$

The geometric genus $p_g(C)$ is defined as $h^0(C, \omega_C)$. To calculate this, consider the standard short exact sequence defining the closed subscheme C:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(-d) \longrightarrow \mathscr{O}_{\mathbb{P}^2} \longrightarrow \mathscr{O}_C \longrightarrow 0.$$

We twist this sequence by $\mathscr{O}_{\mathbb{P}^2}(d-3)$ to obtain:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(-3) \longrightarrow \mathscr{O}_{\mathbb{P}^2}(d-3) \longrightarrow \mathscr{O}_C(d-3) \longrightarrow 0.$$

Now, take the long exact sequence of cohomology. The relevant segment is:

$$H^0(\mathbb{P}^2, \mathscr{O}(d-3)) \longrightarrow H^0(C, \mathscr{O}_C(d-3)) \longrightarrow H^1(\mathbb{P}^2, \mathscr{O}(-3)).$$

We know that $H^1(\mathbb{P}^2, \mathscr{O}(k)) = 0$ for all integers k. Thus, the restriction map on global sections is surjective. Furthermore, $H^0(\mathbb{P}^2, \mathscr{O}(-3)) = 0$, so the map is injective. Therefore, $H^0(C, \omega_C) \cong H^0(\mathbb{P}^2, \mathscr{O}(d-3))$. The dimension of the space of homogenous polynomials of degree k in $n+1$ variables is $\binom{n+k}{k}$. Here $n = 2$ and $k = d-3$.

$$p_g(C) = \binom{2+(d-3)}{d-3} = \binom{d-1}{2} = \frac{(d-1)(d-2)}{2}.$$

This recovers the classical Plücker formula for the genus of a plane curve via strictly scheme-theoretic methods.

9.3 Ext Functors and Sheaf Ext

Definition 28 Global Ext and Sheaf Ext.

Let X be a ringed space and let $\mathsf{Mod}(X)$ denote the category of $\mathscr{O}_X$-modules. Let $\mathscr{F}$ be a fixed $\mathscr{O}_X$-module.

1. **Global Ext Functors:** The functor $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \cdot) : \mathsf{Mod}(X) \to \mathsf{Ab}$ is left exact. Its right derived functors are denoted by $\mathrm{Ext}^i_{\mathscr{O}_X}(\mathscr{F}, \cdot)$. Thus, for any $\mathscr{O}_X$-module $\mathscr{G}$, the group $\mathrm{Ext}^i_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is the i-th cohomology object of the complex $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{I}^\bullet)$, where $\mathscr{I}^\bullet$ is an injective resolution of $\mathscr{G}$.

2. **Sheaf Ext Functors:** The sheaf internal Hom functor $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \cdot) : \mathsf{Mod}(X) \to \mathsf{Mod}(X)$ is also left exact. Its right derived functors are denoted by $\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F}, \cdot)$. These are sheaves of $\mathscr{O}_X$-modules. Explicitly, $\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is the i-th cohomology sheaf of the complex $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{I}^\bullet)$.

Lemma 29 Restriction of Injective Sheaves.

Let $\mathscr{I}$ be an injective object in $\mathsf{Mod}(X)$. For any open subset $U \subseteq X$, the restriction $\mathscr{I}|_U$ is an injective object in $\mathsf{Mod}(U)$.

Proof. Let $j : U \hookrightarrow X$ be the open inclusion map. To show $\mathscr{I}|_U$ is injective, we must show that the functor $\mathrm{Hom}_{\mathscr{O}_U}(\cdot, \mathscr{I}|_U)$ is exact. Let $0 \to \mathcal{A} \to \mathcal{B}$ be an exact sequence of $\mathscr{O}_U$-modules. We apply the "extension by zero" functor $j_!$, which is exact (as it acts on stalks by either identity or zero). Thus, $0 \to j_!\mathcal{A} \to j_!\mathcal{B}$ is exact on X.

By the adjunction between extension by zero and restriction, we have a natural isomorphism:

$$\mathrm{Hom}_{\mathscr{O}_U}(\mathcal{A}, \mathscr{I}|_U) \cong \mathrm{Hom}_{\mathscr{O}_X}(j_!\mathcal{A}, \mathscr{I}).$$

Since $\mathscr{I}$ is injective on X, the functor $\mathrm{Hom}_{\mathscr{O}_X}(\cdot, \mathscr{I})$ is exact. Therefore, the sequence

$$\mathrm{Hom}_{\mathscr{O}_X}(j_!\mathcal{B}, \mathscr{I}) \to \mathrm{Hom}_{\mathscr{O}_X}(j_!\mathcal{A}, \mathscr{I}) \to 0$$

is exact. Via the adjunction, this implies

$$\mathrm{Hom}_{\mathscr{O}_U}(\mathcal{B}, \mathscr{I}|_U) \to \mathrm{Hom}_{\mathscr{O}_U}(\mathcal{A}, \mathscr{I}|_U) \to 0$$

is exact. Hence, $\mathscr{I}|_U$ is injective. □

Proposition 30 Localization of Sheaf Ext.

For any open subset $U \subseteq X$ and any $\mathscr{O}_X$-modules $\mathscr{F}, \mathscr{G}$, there is a natural isomorphism:

$$\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})|_U \cong \mathscr{E}xt^i_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U).$$

Proof. We regard both sides as functors in $\mathscr{G}$ from $\mathsf{Mod}(X)$ to $\mathsf{Mod}(U)$. We verify they satisfy the axioms of a universal δ-functor.

First, for $i = 0$, the isomorphism holds since sheaf Hom commutes with restriction:

$\mathscr{H}om(\mathscr{F},\mathscr{G})|_U \cong \mathscr{H}om(\mathscr{F}|_U,\mathscr{G}|_U)$.

Second, both sides preserve short exact sequences (giving long exact sequences) because they are derived functors (or compositions thereof with the exact restriction functor).

Third, we check effaceability. If $\mathscr{G} = \mathscr{I}$ is injective on X, then $\mathscr{E}xt^i(\mathscr{F},\mathscr{I}) = 0$ for $i > 0$, so the LHS vanishes. By Lemma 29, $\mathscr{I}|_U$ is injective on U, so $\mathscr{E}xt^i(\mathscr{F}|_U,\mathscr{I}|_U) = 0$ for $i > 0$, implying the RHS vanishes.

Since both functors are effaceable δ-functors agreeing at $i = 0$, they are uniquely isomorphic for all i. □

Proposition 31 Ext of the Structure Sheaf.

For any $\mathscr{O}_X$-module $\mathscr{G}$, we have the following characterizations:

1. $\mathscr{E}xt^0_{\mathscr{O}_X}(\mathscr{O}_X,\mathscr{G}) \cong \mathscr{G}$.
2. $\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{O}_X,\mathscr{G}) = 0$ for all $i > 0$.
3. $\mathrm{Ext}^i_{\mathscr{O}_X}(\mathscr{O}_X,\mathscr{G}) \cong H^i(X,\mathscr{G})$ for all $i \geq 0$.

Proof. For parts (1) and (2), consider the functor $T(\cdot) = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X,\cdot)$. There is a canonical isomorphism $T(\mathscr{G}) \cong \mathscr{G}$ defined by $\phi \mapsto \phi(1)$. Thus, T is naturally isomorphic to the identity functor. Since the identity functor is exact, its higher derived functors vanish. Therefore, $\mathscr{E}xt^0 \cong \mathrm{id}$ and $\mathscr{E}xt^i = 0$ for $i > 0$.

For part (3), the global Ext functors are the derived functors of $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X,\cdot)$. Observe that for any sheaf $\mathscr{G}$,

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X,\mathscr{G}) \cong \Gamma(X,\mathscr{G}).$$

Thus, we are computing the right derived functors of the global section functor $\Gamma(X,\cdot)$. By definition, the right derived functors of $\Gamma(X,\cdot)$ are the sheaf cohomology groups $H^i(X,\cdot)$. Hence, $\mathrm{Ext}^i_{\mathscr{O}_X}(\mathscr{O}_X,\mathscr{G}) \cong H^i(X,\mathscr{G})$. □

Proposition 32 Long Exact Sequence in the First Variable.

Let X be a ringed space. Suppose there is a short exact sequence of $\mathscr{O}_X$-modules:

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0.$$

Then for any $\mathscr{O}_X$-module $\mathscr{G}$, there exist long exact sequences of abelian groups and sheaves respectively:

$$\begin{aligned} 0 &\to \mathrm{Hom}(\mathscr{F}'',\mathscr{G}) \to \mathrm{Hom}(\mathscr{F},\mathscr{G}) \to \mathrm{Hom}(\mathscr{F}',\mathscr{G}) \\ &\to \mathrm{Ext}^1(\mathscr{F}'',\mathscr{G}) \to \mathrm{Ext}^1(\mathscr{F},\mathscr{G}) \to \mathrm{Ext}^1(\mathscr{F}',\mathscr{G}) \to \dots \end{aligned}$$

and

$$\begin{aligned} 0 &\to \mathscr{H}om(\mathscr{F}'',\mathscr{G}) \to \mathscr{H}om(\mathscr{F},\mathscr{G}) \to \mathscr{H}om(\mathscr{F}',\mathscr{G}) \\ &\to \mathscr{E}xt^1(\mathscr{F}'',\mathscr{G}) \to \mathscr{E}xt^1(\mathscr{F},\mathscr{G}) \to \mathscr{E}xt^1(\mathscr{F}',\mathscr{G}) \to \dots \end{aligned}$$

Proof. We present the proof for the global Ext functor; the argument for the sheaf $\mathscr{E}xt$ is formally identical.

Let $0 \to \mathscr{G} \to \mathscr{I}^\bullet$ be an injective resolution of $\mathscr{G}$. By the definition of derived functors, the groups $\mathrm{Ext}^i(\cdot, \mathscr{G})$ are computed as the cohomology of the complex $\mathrm{Hom}(\cdot, \mathscr{I}^\bullet)$.

Consider the functor $\mathrm{Hom}_{\mathscr{O}_X}(\cdot, \mathcal{J})$ where $\mathcal{J}$ is an injective $\mathscr{O}_X$-module. By the definition of injectivity, this functor is exact. Therefore, applying $\mathrm{Hom}(\cdot, \mathscr{I}^k)$ (for each $k \geq 0$) to the given short exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ yields a short exact sequence of abelian groups:

$$0 \to \mathrm{Hom}(\mathscr{F}'', \mathscr{I}^k) \to \mathrm{Hom}(\mathscr{F}, \mathscr{I}^k) \to \mathrm{Hom}(\mathscr{F}', \mathscr{I}^k) \to 0.$$

Since this exactness holds for every term in the resolution, we obtain a short exact sequence of complexes:

$$0 \to \mathrm{Hom}(\mathscr{F}'', \mathscr{I}^\bullet) \to \mathrm{Hom}(\mathscr{F}, \mathscr{I}^\bullet) \to \mathrm{Hom}(\mathscr{F}', \mathscr{I}^\bullet) \to 0.$$

Standard homological algebra (specifically the Zig-Zag Lemma or Snake Lemma) implies that a short exact sequence of complexes induces a long exact sequence in cohomology. Since the cohomology groups of these complexes are precisely the Ext groups, the desired result follows immediately. □

Proposition 33 Computation via Locally Free Resolutions.

Let $\mathscr{F}$ be an $\mathscr{O}_X$-module which admits a locally free resolution $\mathcal{L}_\bullet \to \mathscr{F} \to 0$ of finite rank. That is, there is an exact sequence:

$$\cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathscr{F} \to 0,$$

where each $\mathcal{L}_i$ is a locally free $\mathscr{O}_X$-module of finite rank. Then for any $\mathscr{O}_X$-module $\mathscr{G}$, there is a natural isomorphism:

$$\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \cong h^i(\mathscr{H}om_{\mathscr{O}_X}(\mathcal{L}_\bullet, \mathscr{G})).$$

Proof. We prove this by showing that both sides of the isomorphism constitute universal δ-functors in the variable $\mathscr{G}$.

Let $T^i(\mathscr{G})$ denote the right hand side, $h^i(\mathscr{H}om_{\mathscr{O}_X}(\mathcal{L}_\bullet, \mathscr{G}))$. First, we examine the case $i = 0$. The sequence $\mathcal{L}_1 \to \mathcal{L}_0 \to \mathscr{F} \to 0$ is exact. Since the functor $\mathscr{H}om_{\mathscr{O}_X}(\cdot, \mathscr{G})$ is left exact and contravariant, the induced sequence

$$0 \to \mathscr{H}om(\mathscr{F}, \mathscr{G}) \to \mathscr{H}om(\mathcal{L}_0, \mathscr{G}) \to \mathscr{H}om(\mathcal{L}_1, \mathscr{G})$$

is exact. The term $h^0(\mathscr{H}om(\mathcal{L}_\bullet, \mathscr{G}))$ is defined as the kernel of the map between the Hom sheaves of $\mathcal{L}_0$ and $\mathcal{L}_1$. Therefore, we have a natural isomorphism $T^0(\mathscr{G}) \cong \mathscr{H}om(\mathscr{F}, \mathscr{G}) \cong \mathscr{E}xt^0(\mathscr{F}, \mathscr{G})$.

Next, we verify that $T^\bullet$ is a universal δ-functor. It suffices to show that $T^i(\mathscr{G})$ is an effaceable functor for $i > 0$. Let $\mathscr{I}$ be an injective $\mathscr{O}_X$-module. We must show that $T^i(\mathscr{I}) = 0$ for all $i > 0$. Consider the functor $\mathscr{H}om_{\mathscr{O}_X}(\cdot, \mathscr{I})$. As established in previous propositions, since $\mathscr{I}$ is injective, this functor is exact. Applying this exact functor to the exact resolution

$\cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathscr{F} \to 0$ yields an exact sequence of sheaves:

$$0 \to \mathscr{H}om(\mathscr{F}, \mathscr{I}) \to \mathscr{H}om(\mathcal{L}_0, \mathscr{I}) \to \mathscr{H}om(\mathcal{L}_1, \mathscr{I}) \to \dots$$

Since this complex is exact, its cohomology groups vanish for all degrees $i > 0$. Thus $T^i(\mathscr{I}) = 0$.

Since both $(\mathscr{E}xt^i(\mathscr{F}, \cdot))_{i\geq 0}$ and $(T^i(\cdot))_{i\geq 0}$ are universal δ-functors that coincide at $i = 0$, they are naturally isomorphic for all i. □

Lemma 34 Tensor Product of Injective and Locally Free.

Let X be a ringed space. Let $\mathscr{I}$ be an injective $\mathscr{O}_X$-module and let $\mathcal{L}$ be a locally free $\mathscr{O}_X$-module of finite rank. Then the tensor product $\mathcal{L} \otimes_{\mathscr{O}_X} \mathscr{I}$ is an injective $\mathscr{O}_X$-module.

Proof. To prove that $\mathcal{L} \otimes \mathscr{I}$ is injective, it suffices to show that the contravariant functor $\mathrm{Hom}_{\mathscr{O}_X}(\cdot, \mathcal{L} \otimes \mathscr{I})$ is exact. Since $\mathcal{L}$ is locally free of finite rank, it admits a dual locally free sheaf $\mathcal{L}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathcal{L}, \mathscr{O}_X)$. The tensor-hom adjunction yields a natural isomorphism:

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G}, \mathcal{L} \otimes \mathscr{I}) \cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G} \otimes \mathcal{L}^\vee, \mathscr{I}).$$

The functor on the right can be viewed as the composition of the functor $\cdot \otimes \mathcal{L}^\vee$ followed by $\mathrm{Hom}_{\mathscr{O}_X}(\cdot, \mathscr{I})$. Since $\mathcal{L}^\vee$ is locally free, it is flat, so tensoring with it is exact. Since $\mathscr{I}$ is injective, the Hom functor into it is exact. The composition of two exact functors is exact. Consequently, $\mathrm{Hom}_{\mathscr{O}_X}(\cdot, \mathcal{L} \otimes \mathscr{I})$ preserves short exact sequences, implying that $\mathcal{L} \otimes \mathscr{I}$ is injective. □

Proposition 35 Adjunction of Ext and Tensor Product.

Let $\mathcal{L}$ be a locally free sheaf of finite rank with dual $\mathcal{L}^\vee$. For any $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, there are natural isomorphisms for all $i \geq 0$:

$$\mathrm{Ext}^i_{\mathscr{O}_X}(\mathscr{F} \otimes \mathcal{L}, \mathscr{G}) \cong \mathrm{Ext}^i_{\mathscr{O}_X}(\mathscr{F}, \mathcal{L}^\vee \otimes \mathscr{G})$$

and for the sheaf Ext:

$$\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F} \otimes \mathcal{L}, \mathscr{G}) \cong \mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F}, \mathcal{L}^\vee \otimes \mathscr{G}) \cong \mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \otimes \mathcal{L}^\vee.$$

Proof. We fix $\mathscr{F}$ and $\mathcal{L}$ and regard these expressions as functors in the variable $\mathscr{G}$ from $\mathsf{Mod}(X)$ to Ab (or $\mathsf{Mod}(X)$). We apply the theory of universal δ-functors.

For $i = 0$, the isomorphism $\mathrm{Hom}(\mathscr{F} \otimes \mathcal{L}, \mathscr{G}) \cong \mathrm{Hom}(\mathscr{F}, \mathcal{L}^\vee \otimes \mathscr{G})$ is the standard adjunction. Similarly for the internal sheaf Hom.

To show equality for all i, we verify effaceability. If $\mathscr{G} = \mathscr{I}$ is an injective module, then $\mathrm{Ext}^i(\mathscr{F} \otimes \mathcal{L}, \mathscr{I}) = 0$ for $i > 0$. On the right hand side, we consider $\mathrm{Ext}^i(\mathscr{F}, \mathcal{L}^\vee \otimes \mathscr{I})$. By Lemma 34, the sheaf $\mathcal{J} = \mathcal{L}^\vee \otimes \mathscr{I}$ is injective. Therefore, $\mathrm{Ext}^i(\mathscr{F}, \mathcal{J}) = 0$ for $i > 0$.

Since both sides are universal δ-functors agreeing at $i = 0$, they are isomorphic for all i. The second isomorphism for sheaf Ext follows similarly using the flatness of $\mathcal{L}^\vee$. □

Proposition 36 Stalks of Sheaf Ext.

Let X be a Noetherian scheme and let $\mathscr{F}$ be a coherent sheaf on X. Let $\mathscr{G}$ be any $\mathscr{O}_X$-module. For any point $x \in X$, there is a natural isomorphism of $\mathscr{O}_{X,x}$-modules:

$$(\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}))_x \cong \mathrm{Ext}^i_{\mathscr{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x).$$

Proof. The statement is local, so we may restrict our attention to an affine open neighborhood $U = \operatorname{Spec} A$ of x, where A is a Noetherian ring. Since $\mathscr{F}$ is coherent and X is locally Noetherian, $\mathscr{F}|_U$ corresponds to a finitely generated module over A. Thus, there exists a locally free resolution of finite rank on U:

$$\mathcal{L}_\bullet \to \mathscr{F}|_U \to 0.$$

Specifically, we can construct this by taking free modules of finite rank corresponding to generators of the module and its kernels.

According to the proposition on computing Ext via resolutions, we have:

$$\mathscr{E}xt^i_{\mathscr{O}_U}(\mathscr{F}|_U,\mathscr{G}|_U) \cong h^i(\mathscr{H}om_{\mathscr{O}_U}(\mathcal{L}_\bullet,\mathscr{G}|_U)).$$

We now take the stalk at x. The functor taking the stalk is exact and commutes with cohomology. Thus:

$$(\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}))_x \cong (h^i(\mathscr{H}om_{\mathscr{O}_U}(\mathcal{L}_\bullet,\mathscr{G}|_U)))_x \cong h^i((\mathscr{H}om_{\mathscr{O}_U}(\mathcal{L}_\bullet,\mathscr{G}|_U))_x).$$

Since the sheaves $\mathcal{L}_k$ are locally free of finite rank, the stalk of the internal Hom sheaf commutes with the Hom of the stalks:

$$(\mathscr{H}om_{\mathscr{O}_U}(\mathcal{L}_k,\mathscr{G}))_x \cong \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathcal{L}_{k,x},\mathscr{G}_x).$$

Consequently, we obtain:

$$(\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}))_x \cong h^i(\mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathcal{L}_{\bullet,x},\mathscr{G}_x)).$$

Observing that $\mathcal{L}_{\bullet,x} \to \mathscr{F}_x \to 0$ is a free resolution of the $\mathscr{O}_{X,x}$-module $\mathscr{F}_x$ (by the exactness of localization), the cohomology on the right hand side is precisely the definition of $\mathrm{Ext}^i_{\mathscr{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x)$.

Note: This isomorphism fails in general if $\mathscr{F}$ is not coherent (or at least locally finitely presented), as one cannot guarantee a resolution that commutes nicely with Hom and localization. □

Proposition 37 Asymptotic Isomorphism for Very Ample Twists.

Let X be a projective scheme over a Noetherian ring A, and let $\mathscr{O}_X(1)$ be a very ample invertible sheaf on X. Let $\mathscr{F}$ and $\mathscr{G}$ be coherent sheaves on X.

Then there exists an integer n_0 (depending on $\mathscr{F},\mathscr{G},i$) such that for every $n \geq n_0$, the natural map is an isomorphism:

$$\mathrm{Ext}^i_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}(n)) \xrightarrow{\cong} \Gamma(X,\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}(n))).$$

Proof. We proceed by induction on i and the use of locally free resolutions.

Case $i = 0$: By definition, $\mathrm{Ext}^0(\mathscr{F},\mathscr{G}(n)) = \mathrm{Hom}(\mathscr{F},\mathscr{G}(n))$. The global section functor applied to the sheaf Hom yields global Hom: $\Gamma(X,\mathscr{H}om(\mathscr{F},\mathscr{G}(n))) \cong \mathrm{Hom}(\mathscr{F},\mathscr{G}(n))$. This holds for all n without restriction.

Case $\mathscr{F}$ is locally free: If $\mathscr{F}$ is locally free of finite rank, then $\mathscr{E}xt^i(\mathscr{F},\mathscr{G}(n)) = 0$ for all $i > 0$ because locally $\mathscr{F}$ is free and thus projective relative to the structure sheaf. On the global side, we have $\mathrm{Ext}^i(\mathscr{F},\mathscr{G}(n)) \cong H^i(X,\mathscr{F}^\vee \otimes \mathscr{G}(n))$. By Serre's Vanishing Theorem, since $\mathscr{F}^\vee \otimes \mathscr{G}$ is coherent, there exists n_0 such that for all $n \geq n_0$, $H^i(X,\mathscr{F}^\vee \otimes \mathscr{G}(n)) = 0$ for all $i > 0$. Thus, for locally free $\mathscr{F}$, both sides vanish for $i > 0$ and large n, satisfying the isomorphism.

General Case: Since X is projective, any coherent sheaf $\mathscr{F}$ admits a resolution by locally free sheaves of finite rank. We consider a short exact sequence:

$$0 \to \mathcal{K} \to \mathcal{L} \to \mathscr{F} \to 0,$$

where $\mathcal{L}$ is a finite direct sum of twisted structure sheaves $\mathscr{O}_X(m)$ (hence locally free). Since $\mathscr{F}$ and $\mathcal{L}$ are coherent, the kernel $\mathcal{K}$ is also coherent.

We apply the long exact sequence of $\mathrm{Ext}(\cdot,\mathscr{G}(n))$ and $\mathscr{E}xt(\cdot,\mathscr{G}(n))$ to this resolution. For $n \gg 0$, the sheaf $\mathscr{E}xt^1(\mathcal{L},\mathscr{G}(n))$ vanishes. Thus we have an exact sequence of sheaves:

$$0 \to \mathscr{H}om(\mathscr{F},\mathscr{G}(n)) \to \mathscr{H}om(\mathcal{L},\mathscr{G}(n)) \to \mathscr{H}om(\mathcal{K},\mathscr{G}(n)) \to \mathscr{E}xt^1(\mathscr{F},\mathscr{G}(n)) \to 0.$$

Applying the global section functor $\Gamma(X,\cdot)$ to this sequence, exactness is preserved at the first three terms, but Γ is generally only left exact. However, by Serre's Theorem B, for $n \gg 0$, the first cohomology group of the kernel of the map between Hom sheaves vanishes (specifically, $H^1(X,\mathscr{H}om(\mathscr{F},\mathscr{G}(n))) = 0$ because it is a coherent sheaf). Therefore, the sequence of global sections is exact:

$$0 \to \Gamma(\mathscr{H}om(\mathscr{F})) \to \Gamma(\mathscr{H}om(\mathcal{L})) \to \Gamma(\mathscr{H}om(\mathcal{K})) \to \Gamma(\mathscr{E}xt^1(\mathscr{F})) \to 0.$$

Comparing this with the global Ext exact sequence (where $\mathrm{Ext}^1(\mathcal{L},\mathscr{G}(n)) = 0$), and using the 5-lemma (or direct diagram chasing) with the known isomorphisms for $i = 0$, we deduce the isomorphism for $i = 1$:

$$\mathrm{Ext}^1(\mathscr{F},\mathscr{G}(n)) \cong \Gamma(X,\mathscr{E}xt^1(\mathscr{F},\mathscr{G}(n))).$$

For $i > 1$, we observe that $\mathrm{Ext}^i(\mathscr{F},\mathscr{G}(n)) \cong \mathrm{Ext}^{i-1}(\mathcal{K},\mathscr{G}(n))$ and $\mathscr{E}xt^i(\mathscr{F},\mathscr{G}(n)) \cong \mathscr{E}xt^{i-1}(\mathcal{K},\mathscr{G}(n))$ for large n (since $\mathcal{L}$ contributes nothing). By induction on i (replacing $\mathscr{F}$ with $\mathcal{K}$), the result holds for all i. □

The relationship between the global Ext groups (derived from global Hom) and the $\mathscr{E}xt$ sheaves (derived from internal Hom) is often a source of confusion. The following table summarizes their relationships across different geometric contexts.

Note 38 The Local-to-Global Spectral Sequence.

The failure of the isomorphism $\mathrm{Ext}^i \cong \Gamma(\mathscr{E}xt^i)$ in the general global case is measured by

Context	**Module Ext** $\mathrm{Ext}^i_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$	**Sheaf Ext** $\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$	**Isomorphism?**
Affine Scheme	Isomorphic to the A-module $\mathrm{Ext}^i_A(M,N)$.	The quasi-coherent sheaf associated to $\mathrm{Ext}^i_A(M,N)$.	**Yes.** $\Gamma(X,\mathscr{E}xt^i) \cong \mathrm{Ext}^i$.
Stalks	Defined over local ring $\mathscr{O}_{X,x}$: $\mathrm{Ext}^i_{\mathscr{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x)$.	The stalk of the sheaf: $(\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}))_x$.	**Yes.** ($\mathscr{F}$ must be coherent).
Global	Derived functor of global Hom. Depends on global geometry.	A sheaf. $\Gamma(X,\mathscr{E}xt^i)$ captures "local" extensions.	**No.** Via spectral sequence: $H^p(X,\mathscr{E}xt^q) \Rightarrow \mathrm{Ext}^{p+q}$.
Twisted ($n \gg 0$)	Vanishes for $i > 0$ if $\mathscr{F}$ is locally free. Becomes local.	Remains the sheaf $\mathscr{E}xt^i(\dots)$.	**Yes.** (Asymptotic behavior).

Table 9.1: Comparison of Ext Functors and Sheaves

cohomology. There is a spectral sequence:

$$E_2^{p,q} = H^p(X,\mathscr{E}xt^q_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})) \implies \mathrm{Ext}^{p+q}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}).$$

The proposition above essentially states that for $n \gg 0$, $H^p(X,\dots) = 0$ for $p > 0$ (Serre's Vanishing), causing the spectral sequence to collapse to the edge map on the $p = 0$ axis, yielding $\mathrm{Ext}^i \cong H^0(X,\mathscr{E}xt^i) = \Gamma(X,\mathscr{E}xt^i)$.

9.4 Serre Duality Theorem

Readers familiar with differential manifolds will notice that the progression of this chapter runs in high parallel with the theory of manifolds. Indeed, many of the concepts introduced here are algebraic generalizations of classical geometric notions. In modern practice, we no longer strictly distinguish between the following pairs of concepts:

- **Vector Bundle** $\longleftrightarrow$ **Locally Free Sheaf**
- **Line Bundle** $\longleftrightarrow$ **Invertible Sheaf**
- **Cotangent Bundle** $\longleftrightarrow$ **Cotangent Sheaf** $\mathcal{I}/\mathcal{I}^2$
- **Tangent Bundle** $\longleftrightarrow$ **Tangent Sheaf** $(\mathcal{I}/\mathcal{I}^2)^\vee$
- d**-forms** $\longleftrightarrow$ **Exterior Power Sheaf** $\bigwedge^d \Omega_{X/k}$

Furthermore, this parallelism extends to deeper structural correspondences:

- **Smooth Manifold** $\longleftrightarrow$ **Non-singular Variety**
- **Compactness** $\longleftrightarrow$ **Properness / Projectivity**
- **Exterior Derivative** d $\longleftrightarrow$ **The morphism** $d : \mathcal{O}_X \to \Omega^1_{X/k}$

- **Poincaré Duality** $\longleftrightarrow$ **Serre Duality**
- **Integration** $\int_M \omega \longleftrightarrow$ **The Trace Map** $H^n(X, \omega_X) \xrightarrow{\text{tr}} k$

  Note: This analogy reveals how algebraic geometry handles infinitesimal structures. For instance, while differential geometry defines the cotangent bundle via the dual of the tangent space, algebraic geometry captures "first-order variation" through the ideal sheaf of the diagonal $\mathcal{I}$ modulo its square $\mathcal{I}/\mathcal{I}^2$—a purely algebraic construction that recovers the same geometric intuition.

Intuition 39 Serre Duality: A Profound Symmetry in Geometry.

The Serre Duality Theorem is a fundamental and classical result of immense weight in both algebraic and complex geometry. It establishes a profound duality between the cohomology groups of coherent sheaves on projective schemes, with the canonical sheaf and the dualizing sheaf serving as the focal points of this symmetry.

The significance of Serre Duality stems from two sources: first, its aesthetic simplicity paired with mathematical depth; and second, its role as the foundation for numerous pivotal theorems, such as the Residue Theorem and the Riemann-Roch Theorem. It is no exaggeration to say that once Serre Duality is understood, many classical problems transition from formidable challenges to straightforward corollaries.

We begin our exploration with the most fundamental projective scheme: the projective space $\mathbb{P}^n_k$. Duality in the context of projective space is relatively accessible, as its proof relies primarily on the explicit computation of its cohomology groups.

Theorem 40 Serre Duality for P^n.

Let k be a field, and let $X = \mathsf{P}^n_k$ be the projective space of dimension n. Let $\omega_X = \bigwedge^n \Omega_{X/k}$ be the canonical sheaf. Then:

1. There is an isomorphism $H^n(X, \omega_X) \cong k$.
2. For any coherent sheaf $\mathscr{F}$ on X, the natural pairing
$$\operatorname{Hom}(\mathscr{F}, \omega_X) \times H^n(X, \mathscr{F}) \to H^n(X, \omega_X) \cong k$$
is a perfect pairing of finite-dimensional vector spaces over k. Equivalently, the natural map $\theta_{\mathscr{F}} : \operatorname{Hom}(\mathscr{F}, \omega_X) \to H^n(X, \mathscr{F})^\vee$ is an isomorphism.
3. For every integer $i \geq 0$, there is a natural functorial isomorphism
$$\operatorname{Ext}^i(\mathscr{F}, \omega_X) \cong H^{n-i}(X, \mathscr{F})^\vee.$$

Proof. We proceed by analyzing the structure of the canonical sheaf and utilizing the properties of cohomological δ-functors.

First, consider statement (a). Recall the Euler exact sequence for the projective space:
$$0 \to \Omega_{X/k} \to \mathscr{O}_X(-1)^{\oplus(n+1)} \to \mathscr{O}_X \to 0.$$

Taking the highest exterior power yields an isomorphism $\omega_X \cong \mathscr{O}_X(-n-1)$. The cohomology of line bundles on P^n is explicitly known. Specifically, $H^n(X, \mathscr{O}_X(-n-1))$ is dual to $H^0(X, \mathscr{O}_X(0)) \cong k$. Thus, we fix an isomorphism $\eta : H^n(X, \omega_X) \xrightarrow{\sim} k$.

Next, we address statement (b). We aim to show that the map $\theta_{\mathscr{F}} : \mathrm{Hom}(\mathscr{F}, \omega_X) \to H^n(X, \mathscr{F})^\vee$ is an isomorphism for any coherent sheaf $\mathscr{F}$. The proof proceeds in three steps of increasing generality.

Initially, suppose $\mathscr{F} = \mathscr{O}_X(q)$ for some integer q. We have

$$\mathrm{Hom}(\mathscr{O}_X(q), \omega_X) \cong H^0(X, \omega_X(-q)) \cong H^0(X, \mathscr{O}_X(-n-1-q)).$$

On the dual side, the space $H^n(X, \mathscr{O}_X(q))^\vee$ is the dual of the vector space $H^n(X, \mathscr{O}_X(q))$. By the explicit calculation of cohomology on projective space, the pairing between $H^0(X, \mathscr{O}_X(d))$ and $H^n(X, \mathscr{O}_X(-d-n-1))$ is perfect. Setting $d = -n-1-q$, we observe that the result holds for all twisted structure sheaves $\mathscr{O}_X(q)$. Consequently, by the additivity of the functors involved, the isomorphism holds for any sheaf $\mathscr{E}$ which is a finite direct sum of line bundles $\bigoplus \mathscr{O}_X(q_j)$.

For the general case, let $\mathscr{F}$ be an arbitrary coherent sheaf on X. Since $X = \mathsf{P}^n$, every coherent sheaf is a quotient of a direct sum of line bundles. In fact, we can construct a finite presentation:

$$\mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0,$$

where $\mathscr{E}_0$ and $\mathscr{E}_1$ are direct sums of sheaves of the form $\mathscr{O}_X(q)$. Consider the functor $T(\cdot) = \mathrm{Hom}(\cdot, \omega_X)$ and $U(\cdot) = H^n(X, \cdot)^\vee$. The functor T is left-exact. The functor $H^n(X, \cdot)$ is right-exact because $H^{n+1}(X, \cdot) = 0$ for dimensional reasons; consequently, its dual U is left-exact. Applying these functors to the presentation yields the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathscr{F}, \omega_X) \arrow[r] \arrow[d, "\theta_{\mathscr{F}}"] & \mathrm{Hom}(\mathscr{E}_0, \omega_X) \arrow[r] \arrow[d, "\theta_{\mathscr{E}_0}", "\cong"'] & \mathrm{Hom}(\mathscr{E}_1, \omega_X) \arrow[d, "\theta_{\mathscr{E}_1}", "\cong"'] \\
0 \arrow[r] & H^n(X, \mathscr{F})^\vee \arrow[r] & H^n(X, \mathscr{E}_0)^\vee \arrow[r] & H^n(X, \mathscr{E}_1)^\vee
\end{tikzcd}$$

The vertical maps $\theta_{\mathscr{E}_0}$ and $\theta_{\mathscr{E}_1}$ are isomorphisms by our previous step. By the diagram chase (specifically, the truncated version of the Five Lemma or the kernel property), it follows that $\theta_{\mathscr{F}}$ is an isomorphism.

Finally, we prove statement (c). We interpret both sides of the desired isomorphism as contravariant cohomological δ-functors from the category of coherent sheaves $\mathsf{Coh}(X)$ to the category of vector spaces Vec_k. Define $L^i(\mathscr{F}) = \mathrm{Ext}^i(\mathscr{F}, \omega_X)$ and $R^i(\mathscr{F}) = H^{n-i}(X, \mathscr{F})^\vee$. For $i = 0$, we have established $L^0(\mathscr{F}) \cong R^0(\mathscr{F})$ in part (b).

To conclude that $L^i \cong R^i$ for all i, it suffices to show that both sequence of functors are coeffaceable for $i > 0$. A contravariant functor F^i is coeffaceable if for every coherent $\mathscr{F}$, there exists a surjection $\mathscr{E} \twoheadrightarrow \mathscr{F}$ such that $F^i(\mathscr{E}) = 0$ for $i > 0$. By Serre's Theorem B, for any coherent sheaf $\mathscr{F}$, we can find a surjection $\mathscr{E} = \bigoplus_{j=1}^N \mathscr{O}_X(-q_j) \twoheadrightarrow \mathscr{F}$ for sufficiently large integers $q_j \gg 0$. Let us check the vanishing conditions for this $\mathscr{E}$.

For the functor L^i, we examine $\mathrm{Ext}^i(\mathscr{O}_X(-q), \omega_X)$. Since $\mathscr{O}_X(-q)$ is locally free,

$$\mathrm{Ext}^i(\mathscr{O}_X(-q), \omega_X) \cong H^i(X, \omega_X \otimes \mathscr{O}_X(q)) \cong H^i(X, \mathscr{O}_X(q-n-1)).$$

For $q \gg 0$, specifically $q > n+1$, the degree $d = q - n - 1$ is positive. By the cohomology of projective space, $H^i(X, \mathscr{O}_X(d)) = 0$ for all $i > 0$. Thus L^i is coeffaceable.

For the functor R^i, we examine $H^{n-i}(X, \mathscr{O}_X(-q))^\vee$. For $i > 0$, we are looking at cohomology in degree strictly less than n. Since $q \gg 0$, $-q$ is very negative. The only non-vanishing cohomology for $\mathscr{O}_X(-q)$ with negative degree occurs at dimension n. Thus, for $n - i < n$, we have $H^{n-i}(X, \mathscr{O}_X(-q)) = 0$. Consequently, R^i is also coeffaceable.

By the universality of coeffaceable δ-functors, the isomorphism at $i = 0$ extends uniquely to a natural isomorphism for all $i \geq 0$. ☐

To establish the Serre Duality Theorem for general projective schemes, it is necessary to introduce the notions of the dualizing sheaf and the trace map. In fact, the existence and uniqueness (up to isomorphism) of the dualizing sheaf can be guaranteed as long as the scheme is proper. However, for most practical applications and foundational developments, we typically work within the more specific framework of projective schemes.

Definition 41 Dualizing Sheaf and Trace Morphism.

Let k be a field, and let X be a proper scheme of dimension n over k. A **dualizing sheaf** for X is a coherent sheaf ω_X° on X, equipped with a linear map called the **trace morphism**

$$t : H^n(X, \omega_X^\circ) \to k,$$

such that for all coherent sheaves $\mathscr{F}$ on X, the natural pairing

$$\mathrm{Hom}_X(\mathscr{F}, \omega_X^\circ) \times H^n(X, \mathscr{F}) \longrightarrow H^n(X, \omega_X^\circ) \xrightarrow{t} k$$

induces an isomorphism of vector spaces

$$\theta_{\mathscr{F}} : \mathrm{Hom}_X(\mathscr{F}, \omega_X^\circ) \xrightarrow{\sim} H^n(X, \mathscr{F})^\vee.$$

Intuition 42 Representability.

In categorical terms, the definition states that the pair (ω_X°, t) represents the contravariant functor $F : \mathsf{Coh}(X)^{\mathrm{op}} \to \mathsf{Vec}_k$ defined by $\mathscr{F} \mapsto H^n(X, \mathscr{F})^\vee$. The trace map t corresponds to the identity element in $\mathrm{Hom}(\omega_X^\circ, \omega_X^\circ)$ under the induced isomorphism.

Next, we establish the existence and uniqueness of the dualizing sheaf and the trace map by ensuring they satisfy a specific universal property.

Proposition 43 Uniqueness of the Dualizing Sheaf.

Let X be a proper scheme over k. If a dualizing sheaf exists, it is unique up to unique isomorphism.

More precisely, if (ω°, t) and (ω', t') are two pairs satisfying the definition of a dualizing sheaf, then there exists a unique isomorphism $\varphi : \omega^\circ \xrightarrow{\sim} \omega'$ such that the trace maps are compatible, i.e., $t' \circ H^n(\varphi) = t$.

Proof. The proof relies on the universal property inherent in the definition of the dualizing sheaf.

Suppose (ω°, t) and (ω', t') are two such pairs. Since (ω', t') is a dualizing pair, we have a functorial isomorphism for any coherent sheaf $\mathscr{F}$:

$$\theta'_{\mathscr{F}} : \mathrm{Hom}_X(\mathscr{F}, \omega') \xrightarrow{\sim} H^n(X, \mathscr{F})^\vee.$$

We apply this specifically to the sheaf $\mathscr{F} = \omega^\circ$. The dual space $H^n(X, \omega^\circ)^\vee$ contains a distinguished element, namely the trace map t associated with the first pair. Via the isomorphism θ'_{ω°, there corresponds a unique morphism $\varphi : \omega^\circ \to \omega'$ such that

$$\theta'_{\omega^\circ}(\varphi) = t.$$

By the definition of the pairing, this equality is explicitly written as the commutativity of the following diagram:

$$\begin{tikzcd}
H^n(X, \omega^\circ) \arrow[r, "H^n(\varphi)"] \arrow[dr, "t"'] & H^n(X, \omega') \arrow[d, "t'"] \\
 & k
\end{tikzcd}$$

Thus, $t' \circ H^n(\varphi) = t$.

To show that φ is an isomorphism, we invoke the symmetric argument. Since (ω°, t) is also a dualizing pair, there exists a unique morphism $\psi : \omega' \to \omega^\circ$ such that $t \circ H^n(\psi) = t'$.

Consider the composition $\psi \circ \varphi : \omega^\circ \to \omega^\circ$. The induced map on cohomology satisfies:

$$t \circ H^n(\psi \circ \varphi) = t \circ H^n(\psi) \circ H^n(\varphi) = t' \circ H^n(\varphi) = t.$$

However, the identity map $\mathrm{id}_{\omega^\circ}$ also satisfies the condition $t \circ H^n(\mathrm{id}_{\omega^\circ}) = t$. Since the morphism in $\mathrm{Hom}(\omega^\circ, \omega^\circ)$ mapping to t under θ_{ω° is unique, we must have $\psi \circ \varphi = \mathrm{id}_{\omega^\circ}$.

Similarly, $\varphi \circ \psi = \mathrm{id}_{\omega'}$. Therefore, φ is an isomorphism, and it is the unique one compatible with the trace maps. □

To complete the proof of the Serre Duality Theorem for projective schemes, two key lemmas are required. The first is somewhat technical: it states that for a closed subscheme X of codimension r in projective space $\mathbb{P}^N_k$, the Ext sheaves $\mathscr{E}xt^i_{\mathscr{O}_{\mathbb{P}}}(\mathscr{O}_X, \omega_{\mathbb{P}})$ vanish for all $i < r$.

Lemma 44 Vanishing of Lower Extension Sheaves.

Let $P = \mathsf{P}^N_k$ be the projective space over a field k, and let $X \subset P$ be a closed subscheme of codimension r. Let ω_P be the canonical sheaf of P. Then:

$$\mathscr{E}xt^i_P(\mathscr{O}_X, \omega_P) = 0 \quad \text{for all } i < r.$$

Proof. The sheaf $\mathscr{F}^i = \mathscr{E}xt^i_P(\mathscr{O}_X, \omega_P)$ is coherent on P (since $\mathscr{O}_X$ and ω_P are coherent). To prove $\mathscr{F}^i = 0$, it suffices to show that its space of global sections vanishes after twisting by a sufficiently large integer q, i.e., $\Gamma(P, \mathscr{F}^i(q)) = 0$ for $q \gg 0$ (Serre's Vanishing Theorem).

Using the properties of the local Ext sheaf and its relation to the global Ext group, we have for $q \gg 0$:

$$\Gamma(P, \mathscr{F}^i(q)) \cong \operatorname{Ext}^i_P(\mathscr{O}_X, \omega_P(q)).$$

By the Serre Duality on the ambient projective space P, there is a perfect pairing (isomorphism):

$$\operatorname{Ext}^i_P(\mathscr{O}_X, \omega_P(q)) \cong H^{N-i}(P, \mathscr{O}_X(-q))^\vee.$$

Now, consider the cohomology group $H^{N-i}(P, \mathscr{O}_X(-q))$. The sheaf $\mathscr{O}_X(-q)$ is supported on X. Since the cohomology of a sheaf depends only on its support, we have an isomorphism:

$$H^{N-i}(P, \mathscr{O}_X(-q)) \cong H^{N-i}(X, \mathscr{O}_X(-q)).$$

Let $n = \dim X = N - r$. The condition $i < r$ implies $N - i > N - r = n$. Since the cohomological dimension of a Noetherian scheme is bounded by its dimension (Grothendieck's Vanishing Theorem), $H^k(X, \mathscr{G}) = 0$ for any $k > n$ and any coherent sheaf $\mathscr{G}$. Therefore, $H^{N-i}(X, \mathscr{O}_X(-q)) = 0$ for all $i < r$. Consequently, $\Gamma(P, \mathscr{F}^i(q)) = 0$, which implies $\mathscr{E}xt^i_P(\mathscr{O}_X, \omega_P) = 0$ for $i < r$. □

The second lemma is more structural in nature: we define the dualizing sheaf specifically as $\mathscr{E}xt^r_{\mathscr{O}_\mathbb{P}}(\mathscr{O}_X, \omega_\mathbb{P})$. This definition then induces a functorial isomorphism.

Lemma 45 The Isomorphism for the Dualizing Sheaf Candidate.

With the same hypotheses as in Lemma 44, define the sheaf $\omega_X^\circ := \mathscr{E}xt^r_P(\mathscr{O}_X, \omega_P)$. Then for any $\mathscr{O}_X$-module $\mathscr{F}$, there is a functorial isomorphism:

$$\operatorname{Hom}_X(\mathscr{F}, \omega_X^\circ) \cong \operatorname{Ext}^r_P(\mathscr{F}, \omega_P).$$

Proof. We proceed by explicitly constructing the Ext groups using an injective resolution.

Let $0 \to \omega_P \to \mathscr{J}^\bullet$ be an injective resolution of ω_P in the category of $\mathscr{O}_P$-modules, denoted $\mathsf{Mod}(P)$. By definition, the group $\operatorname{Ext}^r_P(\mathscr{F}, \omega_P)$ is the r-th cohomology group of the complex $\operatorname{Hom}_P(\mathscr{F}, \mathscr{J}^\bullet)$.

Since $\mathscr{F}$ is an $\mathscr{O}_X$-module, the action of $\mathscr{O}_P$ on $\mathscr{F}$ factors through $\mathscr{O}_X = \mathscr{O}_P/\mathscr{I}_X$. Consequently, any $\mathscr{O}_P$-linear morphism from $\mathscr{F}$ to an $\mathscr{O}_P$-module $\mathscr{M}$ must land in the submodule of $\mathscr{M}$ annihilated by $\mathscr{I}_X$. This submodule is isomorphic to $\mathscr{H}om_P(\mathscr{O}_X, \mathscr{M})$. Therefore, we have a natural isomorphism of complexes:

$$\operatorname{Hom}_P(\mathscr{F}, \mathscr{J}^\bullet) \cong \operatorname{Hom}_X(\mathscr{F}, \mathscr{H}om_P(\mathscr{O}_X, \mathscr{J}^\bullet)).$$

Let us define the complex of sheaves on X by $\mathscr{I}^\bullet := \mathscr{H}om_P(\mathscr{O}_X, \mathscr{J}^\bullet)$.

We claim two properties for this complex $\mathscr{I}^\bullet$:

1. Each term $\mathscr{I}^k$ is an injective $\mathscr{O}_X$-module.
2. The cohomology sheaves satisfy $h^k(\mathscr{I}^\bullet) = 0$ for all $k < r$.

Proof of (1): For any $\mathscr{O}_X$-module $\mathscr{G}$, we have the adjunction $\mathrm{Hom}_X(\mathscr{G}, \mathscr{I}^k) = \mathrm{Hom}_X(\mathscr{G}, \mathscr{H}om_P(\mathscr{O}_X, \mathscr{J}^k)) \cong \mathrm{Hom}_P(\mathscr{G}, \mathscr{J}^k)$. Since $\mathscr{J}^k$ is injective in $\mathsf{Mod}(P)$, the functor $\mathrm{Hom}_P(\cdot, \mathscr{J}^k)$ is exact. Thus $\mathrm{Hom}_X(\cdot, \mathscr{I}^k)$ is exact, implying $\mathscr{I}^k$ is an injective $\mathscr{O}_X$-module.

Proof of (2): The cohomology of the complex $\mathscr{I}^\bullet$ is, by definition, the sheaf Ext:

$$h^k(\mathscr{I}^\bullet) = \mathscr{E}xt^k_P(\mathscr{O}_X, \omega_P).$$

By Lemma 44, we know explicitly that $\mathscr{E}xt^k_P(\mathscr{O}_X, \omega_P) = 0$ for all $k < r$.

Now, consider the structure of the complex $\mathscr{I}^\bullet$:

$$0 \longrightarrow \mathscr{I}^0 \longrightarrow \mathscr{I}^1 \longrightarrow \cdots \longrightarrow \mathscr{I}^{r-1} \xrightarrow{d^{r-1}} \mathscr{I}^r \longrightarrow \dots$$

Since the cohomology vanishes for $k < r$, this sequence is exact up to step r. Because each term $\mathscr{I}^k$ is injective, this exact sequence is actually **split exact**. This allows us to decompose the complex into a direct sum of two complexes of injective modules, $\mathscr{I}^\bullet = \mathscr{I}'^\bullet \oplus \mathscr{I}''^\bullet$, where:

- $\mathscr{I}'^\bullet$ is the complex $0 \to \mathscr{I}^0 \to \cdots \to \mathscr{I}^{r-1} \to \mathrm{im}(d^{r-1}) \to 0$, which is exact (acyclic).
- $\mathscr{I}''^\bullet$ is the complex $0 \to \cdots \to 0 \to \ker(d^r) \to \mathscr{I}^r \to \mathscr{I}^{r+1} \to \dots$, shifted appropriately.

Specifically, at index r, the cohomology is determined by the kernel of the map from $\mathscr{I}^r$. Let $Z^r = \ker(d^r : \mathscr{I}^r \to \mathscr{I}^{r+1})$. The cohomology group we seek is:

$$\mathrm{Ext}^r_P(\mathscr{F}, \omega_P) = h^r(\mathrm{Hom}_X(\mathscr{F}, \mathscr{I}^\bullet)).$$

Because of the splitting, applying $\mathrm{Hom}_X(\mathscr{F}, \cdot)$ preserves the exactness of the lower-degree part. The cohomology at degree r is simply isomorphic to the maps into the kernel of the r-th boundary map (modulo the image from $r-1$, which splits off).

By definition, the sheaf $\omega^\circ_X = \mathscr{E}xt^r_P(\mathscr{O}_X, \omega_P)$ is exactly the r-th cohomology sheaf of $\mathscr{I}^\bullet$, which is $Z^r/\mathrm{im}(d^{r-1})$. Since the complex splits, we can identify ω°_X effectively as the "beginning" of the non-trivial part of the resolution. Thus, calculating the cohomology of $\mathrm{Hom}_X(\mathscr{F}, \mathscr{I}^\bullet)$ at index r yields exactly:

$$\mathrm{Hom}_X(\mathscr{F}, \ker(d^r)/\mathrm{im}(d^{r-1})) \cong \mathrm{Hom}_X(\mathscr{F}, \omega^\circ_X).$$

This establishes the desired isomorphism. □

The following lemma formally asserts the existence of a dualizing sheaf on a projective scheme and provides an explicit construction for it; specifically, it is the $\mathcal{E}\S\sqcup$ sheaf defined above.

Proposition 46 Existence of Dualizing Sheaf for Projective Schemes.

Let X be a projective scheme over a field k. Then X possesses a dualizing sheaf.

Proof. Since X is projective, we can embed it as a closed subscheme $X \hookrightarrow P = \mathbb{P}^N_k$ for some N. Let $r = N - \dim X$ be the codimension. Define $\omega^\circ_X := \mathscr{E}xt^r_P(\mathscr{O}_X, \omega_P)$. We claim that ω°_X is the dualizing sheaf for X.

For any coherent sheaf $\mathscr{F}$ on X, we treat it as a sheaf on P via the inclusion i_*. Applying Lemma 45, we have:

$$\mathrm{Hom}_X(\mathscr{F}, \omega^\circ_X) \cong \mathrm{Ext}^r_P(\mathscr{F}, \omega_P).$$

Now we invoke the Serre Duality Theorem for the projective space P (Theorem 40). Since $\mathscr{F}$ is coherent on P, we have a perfect pairing:

$$\mathrm{Ext}^r_P(\mathscr{F}, \omega_P) \cong H^{N-r}(P, \mathscr{F})^\vee.$$

Since $\mathscr{F}$ is supported on X, its cohomology on P is identical to its cohomology on X. With $n = \dim X = N - r$, we have:

$$H^{N-r}(P, \mathscr{F}) \cong H^n(X, \mathscr{F}).$$

Combining these isomorphisms yields:

$$\mathrm{Hom}_X(\mathscr{F}, \omega^\circ_X) \cong H^n(X, \mathscr{F})^\vee.$$

To complete the structure of the dualizing sheaf, we must define the trace morphism t. By setting $\mathscr{F} = \omega^\circ_X$ in the isomorphism above, we obtain:

$$\mathrm{Hom}_X(\omega^\circ_X, \omega^\circ_X) \xrightarrow{\sim} H^n(X, \omega^\circ_X)^\vee.$$

The identity map $\mathrm{id} \in \mathrm{Hom}_X(\omega^\circ_X, \omega^\circ_X)$ corresponds to a linear functional $t \in H^n(X, \omega^\circ_X)^\vee$, which we identify as the map $t : H^n(X, \omega^\circ_X) \to k$.

By the functoriality of the constructed isomorphisms, the pair (ω°_X, t) satisfies the definition of a dualizing sheaf. □

Now that the preliminary groundwork is complete, we proceed to formally state and prove the **Serre Duality Theorem for general projective schemes**.

Theorem 47 Serre Duality for Projective Schemes.

Let X be a projective scheme of dimension n over an algebraically closed field k. Let ω°_X be a dualizing sheaf on X, and let $\mathscr{O}_X(1)$ be a very ample sheaf. Then:

(a) For all integers $i \geq 0$ and every coherent sheaf $\mathscr{F}$ on X, there exist natural functorial maps

$$\theta^i : \mathrm{Ext}^i_X(\mathscr{F}, \omega^\circ_X) \longrightarrow H^{n-i}(X, \mathscr{F})^\vee,$$

such that θ^0 is the isomorphism induced by the definition of the dualizing sheaf.

(b) The following conditions are equivalent:

  (i) X is Cohen-Macaulay and equidimensional.

  (ii) For any locally free sheaf $\mathscr{F}$ on X, the intermediate cohomology of sufficiently negative twists vanishes:

$$H^i(X, \mathscr{F}(-q)) = 0 \quad \text{for all } i < n \text{ and } q \gg 0.$$

(iii) The maps θ^i defined in (a) are isomorphisms for all $i \geq 0$ and all coherent sheaves $\mathscr{F}$.

Proof. **Proof of (a): Construction via Universal δ-Functors.** We interpret both sides of the desired map as contravariant δ-functors from $\mathsf{Coh}(X)$ to Vec_k. Let $T^i(\mathscr{F}) = \mathrm{Ext}^i_X(\mathscr{F}, \omega^\circ_X)$ and $U^i(\mathscr{F}) = H^{n-i}(X, \mathscr{F})^\vee$.

The sequence $\{T^i\}_{i\geq 0}$ forms a universal coeffaceable δ-functor. To see this, recall that for any coherent sheaf $\mathscr{F}$, there exists a surjection from a sheaf $\mathscr{E}$ which is a direct sum of very negative line bundles, $\mathscr{E} = \bigoplus \mathcal{O}_X(-q_j) \twoheadrightarrow \mathscr{F}$ with $q_j \gg 0$. For $i > 0$, we have:

$$T^i(\mathcal{O}_X(-q)) = \mathrm{Ext}^i(\mathcal{O}_X(-q), \omega^\circ_X) \cong H^i(X, \omega^\circ_X(q)).$$

By Serre's Vanishing Theorem, $H^i(X, \omega^\circ_X(q)) = 0$ for $q \gg 0$ and $i > 0$. Thus, T^i vanishes on sufficiently negative vector bundles, proving it is coeffaceable and hence universal.

The sequence $\{U^i\}_{i\geq 0}$ is also a δ-functor (derived from the long exact sequence of cohomology). By the definition of the dualizing sheaf, we are given an isomorphism of functors $\theta^0 : T^0 \xrightarrow{\sim} U^0$. By the universality of $T^\bullet$, there exists a unique morphism of δ-functors $\theta^\bullet : T^\bullet \longrightarrow U^\bullet$ extending θ^0.

Proof of (b): Equivalence of Conditions. We establish the cycle of implications (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i).

Implication (i) $\implies$ (ii). Assume X is Cohen-Macaulay and equidimensional of dimension n. We embed X as a closed subscheme into a projective space $P = \mathsf{P}^N_k$ via the very ample sheaf $\mathcal{O}_X(1)$. Let $\mathscr{F}$ be a locally free sheaf on X. To analyze the cohomology $H^i(X, \mathscr{F}(-q))$, we employ Serre Duality on the ambient space P. Regarding $\mathscr{F}$ as a sheaf on P (via i_*), we have for any i:

$$H^i(X, \mathscr{F}(-q)) \cong \mathrm{Ext}^{N-i}_P(i_*\mathscr{F}, \omega_P(q))^\vee.$$

For $q \gg 0$, the global Ext group is isomorphic to the global sections of the local Ext sheaf:

$$\mathrm{Ext}^k_P(i_*\mathscr{F}, \omega_P(q)) \cong \Gamma(P, \mathscr{E}xt^k_P(i_*\mathscr{F}, \omega_P)(q)).$$

Thus, proving the vanishing of cohomology for $i < n$ is equivalent to proving the vanishing of the sheaf $\mathscr{E}xt^k_P(i_*\mathscr{F}, \omega_P)$ for $k > N - n$.

Let $x \in X$ be a closed point. Let $A = \mathcal{O}_{P,x}$ be the local ring of the ambient space (a regular local ring of dimension N) and let $B = \mathcal{O}_{X,x}$ be the local ring of X. Since $\mathscr{F}$ is locally free on X, $\mathscr{F}_x$ is a free B-module. Therefore, the stalk of the Ext sheaf is:

$$\mathscr{E}xt^k_P(i_*\mathscr{F}, \omega_P)_x \cong \mathrm{Ext}^k_A(\mathscr{F}_x, A) \cong \mathrm{Ext}^k_A(B, A) \otimes_B \mathscr{F}_x.$$

We now invoke the **Auslander-Buchsbaum formula** over the regular local ring A. Since X is Cohen-Macaulay, B is a Cohen-Macaulay module over A. The formula states:

$$\mathrm{pd}_A(B) + \mathrm{depth}_A(B) = \mathrm{depth}(A) = \dim(A) = N.$$

Since X is Cohen-Macaulay of dimension n, $\mathrm{depth}_A(B) = \mathrm{depth}(B) = \dim(B) = n$. Hence, the projective dimension is $\mathrm{pd}_A(B) = N - n$. This implies that $\mathrm{Ext}^k_A(B, A) = 0$ for all

$k > N-n$. Returning to the indices, if $i < n$, then $N-i > N-n$, so the Ext group vanishes. Consequently, $H^i(X, \mathscr{F}(-q)) = 0$.

Implication (ii) $\implies$ (iii). We have already established that the map $\theta^i : \mathrm{Ext}^i_X(\mathscr{F}, \omega^\circ_X) \to H^{n-i}(X, \mathscr{F})^\vee$ comes from the universality of the source δ-functor. To show it is an isomorphism, it suffices (by the standard criterion for morphisms of δ-functors) to show that the target functor $U^i(\cdot) = H^{n-i}(X, \cdot)^\vee$ is also coeffaceable for $i > 0$.

Let $\mathscr{F}$ be any coherent sheaf. We choose a surjection $\mathscr{E} \twoheadrightarrow \mathscr{F}$ where $\mathscr{E}$ is a direct sum of $\mathscr{O}(-q)$ with $q \gg 0$. We must check if $U^i(\mathscr{E}) = 0$ for $i > 0$.

$$U^i(\mathscr{E}) = H^{n-i}(X, \mathscr{E})^\vee.$$

If $i > 0$, then $n-i < n$. By hypothesis (ii), $H^{n-i}(X, \mathscr{O}(-q)) = 0$ for $q \gg 0$. Since cohomology commutes with finite direct sums, $H^{n-i}(X, \mathscr{E}) = 0$. Thus, the target functor is coeffaceable, and θ^i is an isomorphism for all i.

Implication (iii) $\implies$ (ii). Assume the duality isomorphism holds. For $\mathscr{F}$ locally free,

$$H^i(X, \mathscr{F}(-q)) \cong \mathrm{Ext}^{n-i}_X(\mathscr{F}(-q), \omega^\circ_X)^\vee.$$

Using the local freeness of $\mathscr{F}$, the Ext term simplifies to a cohomology group:

$$\mathrm{Ext}^{n-i}_X(\mathscr{F}(-q), \omega^\circ_X) \cong H^{n-i}(X, \mathscr{F}^\vee \otimes \omega^\circ_X(q)).$$

Let $j = n-i$. If $i < n$, then $j > 0$. For $q \gg 0$, Serre Vanishing ensures that $H^j(X, \mathscr{G}(q)) = 0$ for any coherent $\mathscr{G}$ and $j > 0$. Thus the term vanishes, yielding (ii).

Conclusion (ii) $\implies$ (i). We reverse the homological algebra argument used in the first step. Hypothesis (ii) implies that $\Gamma(P, \mathscr{E}xt^k_P(\mathscr{O}_X, \omega_P)(q)) = 0$ for $k > N-n$ and $q \gg 0$, which implies the sheaf $\mathscr{E}xt^k_P(\mathscr{O}_X, \omega_P) = 0$. At a local ring $A = \mathscr{O}_{P,x}$, this means $\mathrm{Ext}^k_A(B, A) = 0$ for $k > N-n$. This vanishing implies $\mathrm{pd}_A(B) \le N-n$. Applying Auslander-Buchsbaum again:

$$\mathrm{depth}(B) = N - \mathrm{pd}_A(B) \ge N - (N-n) = n.$$

Since $\dim(B) \le n$ (as $\dim X = n$), we must have $\mathrm{depth}(B) = \dim(B) = n$. Thus $\mathscr{O}_{X,x}$ is Cohen-Macaulay. Since this holds for all closed points, X is Cohen-Macaulay. □

We now derive a series of immediate corollaries from this fundamental duality theorem:

Duality for Vector Bundles: The first corollary provides a direct dual relationship between the cohomology groups of vector bundles (locally free sheaves).

Corollary 48 Duality for Vector Bundles.

Let X be a projective Cohen-Macaulay scheme of equidimension n over k. Let ω°_X be the dualizing sheaf. Then for any locally free sheaf $\mathscr{F}$ on X, there are natural functorial isomorphisms:

$$H^i(X, \mathscr{F}) \cong H^{n-i}(X, \mathscr{F}^\vee \otimes \omega^\circ_X)^\vee.$$

Proof. We rely on the general duality theorem for projective schemes (Theorem 47). Since X is Cohen-Macaulay, condition (iii) of the theorem holds. This grants us a natural isomorphism for any coherent sheaf $\mathscr{F}$:

$$H^i(X,\mathscr{F}) \cong \operatorname{Ext}_X^{n-i}(\mathscr{F},\omega_X^\circ)^\vee \cong H^{n-i}(X,\mathscr{F}^\vee \otimes \omega_X^\circ)^\vee.$$

□

Vanishing on Normal Projective Schemes: Furthermore, if X is a **normal** projective scheme of dimension $n \geq 2$, the first cohomology group of any vector bundle will vanish after being **sufficiently twisted**.

Corollary 49 Enriques-Severi-Zariski Lemma.

Let X be a normal projective scheme of dimension ≥ 2 over k. Then for any locally free sheaf $\mathscr{F}$ on X, we have the vanishing of the first cohomology group for sufficiently negative twists:

$$H^1(X,\mathscr{F}(-q)) = 0 \quad \text{for } q \gg 0.$$

Proof. The normality of X implies the geometric condition (S_2) according to Serre's Criterion for Normality. Explicitly, for every point $x \in X$, the depth of the local ring satisfies:

$$\operatorname{depth}(\mathscr{O}_{X,x}) \geq \min(2, \dim \mathscr{O}_{X,x}).$$

Since X is a projective scheme of dimension at least 2, for any closed point $x \in X$, we have $\dim \mathscr{O}_{X,x} = \dim X \geq 2$. Consequently, $\operatorname{depth}(\mathscr{O}_{X,x}) \geq 2$.

Recall the argument utilized in the proof of Theorem 47, specifically the implication (i) $\Longrightarrow$ (ii). The vanishing of cohomology $H^i(X,\mathscr{F}(-q))$ for $q \gg 0$ is controlled by the depth of $\mathscr{F}$ (which equals the depth of $\mathscr{O}_{X,x}$ since $\mathscr{F}$ is locally free) via the relation depth $\geq n-i$. To ensure $H^1(X,\mathscr{F}(-q)) = 0$, we require the vanishing of the dual Ext group, which corresponds to the condition $\operatorname{depth}(\mathscr{O}_{X,x}) > \dim X - (n-1) = 1$. Since $\operatorname{depth}(\mathscr{O}_{X,x}) \geq 2$, this condition is satisfied. Thus, the first cohomology group vanishes asymptotically. □

Connectedness of Support: Finally, if X is **integral**, we can discuss divisors. This corollary asserts that the support of any effective ample divisor (or its associated line bundle) is necessarily connected.

Corollary 50 Connectedness of Ample Divisors.

Let X be an integral, normal projective variety of dimension ≥ 2 over an algebraically closed field k. Let Y be a closed subset of codimension 1 which is the support of an effective ample divisor. Then Y is connected.

Proof. By the definition of ampleness, some multiple D of the divisor supported on Y corresponds to a very ample invertible sheaf $\mathscr{O}_X(1)$. For any integer $m > 0$, let Y_m denote the scheme defined by the sheaf of ideals $\mathscr{O}_X(-m)$. The support of Y_m is exactly Y. Consider the short exact sequence of sheaves:

$$0 \longrightarrow \mathscr{O}_X(-m) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_{Y_m} \longrightarrow 0.$$

Taking the long exact sequence of cohomology, we focus on the initial terms:

$$0 \longrightarrow H^0(X, \mathcal{O}_X(-m)) \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(Y_m, \mathcal{O}_{Y_m}) \longrightarrow H^1(X, \mathcal{O}_X(-m)) \longrightarrow \dots$$

For $m \gg 0$, $H^0(X, \mathcal{O}_X(-m)) = 0$ because $\mathcal{O}_X(-m)$ has no global sections (negative degree). Furthermore, by Corollary 49, since X is normal and $\dim X \geq 2$, we have $H^1(X, \mathcal{O}_X(-m)) = 0$. This implies an isomorphism $H^0(X, \mathcal{O}_X) \cong H^0(Y_m, \mathcal{O}_{Y_m})$.

Since X is a projective variety over an algebraically closed field k, $H^0(X, \mathcal{O}_X) \cong k$. Therefore, $H^0(Y_m, \mathcal{O}_{Y_m}) \cong k$. The dimension of the space of global sections of the structure sheaf counts the number of connected components (weighted by local Artinian factors if non-reduced, but here we strictly look at the number of idempotents). Specifically, if Y were disconnected, say $Y = Y' \cup Y''$ with Y', Y'' disjoint and non-empty, then the sheaf $\mathcal{O}_{Y_m}$ would decompose, and $H^0(Y_m, \mathcal{O}_{Y_m})$ would contain at least $k \oplus k$, having dimension ≥ 2. The fact that the dimension is 1 forces Y_m (and hence its support Y) to be connected. □

Note 51 Irreducibility via Bertini.

This corollary significantly strengthens Bertini's Theorem. Recall that Bertini's Theorem asserts that for a generic hyperplane section H of a nonsingular variety $X \subset \mathbb{P}^n$, the intersection $Y = X \cap H$ is nonsingular. However, "nonsingular" does not structurally imply "irreducible"; a disjoint union of two smooth varieties is still a smooth variety.

If $\dim X \geq 2$, the corollary above ensures that Y is *connected*. In the category of schemes, a connected and regular (nonsingular) scheme is necessarily integral (and thus irreducible). *Reasoning:* A regular local ring is an integral domain. If a connected scheme is locally integral, it cannot be the union of two distinct closed irreducible components, as their intersection would lie in the singular locus (which is empty). Therefore, for dimension ≥ 2, a generic hyperplane section is not just smooth, but an irreducible smooth variety.

To further develop our duality theory, we aim to derive a formula for **complete intersections** in projective space that is more intuitive and computationally tractable than the general $\mathcal{E}\S\sqcup$ sheaf. To achieve this, we must first introduce an essential tool from commutative algebra: the **Koszul Complex**.

Definition 52 Koszul Complex.

Let A be a ring and let $f_1, \dots, f_r \in A$ be a sequence of elements. Let M be an A-module. We define the **Koszul complex**, denoted $K_\bullet(f_1, \dots, f_r; M)$, as follows. Let L be a free A-module of rank r with basis $e_1, \dots, e_r$. We form the exterior algebra $\bigwedge^\bullet L$. The complex terms are $K_p = \bigwedge^p L \otimes_A M$. The boundary map $d : K_p \to K_{p-1}$ is defined by the contraction with the vector $(f_1, \dots, f_r)$. Explicitly, on basis elements:

$$d(e_{i_1} \wedge \dots \wedge e_{i_p} \otimes m) = \sum_{j=1}^{p} (-1)^{j-1} f_{i_j} (e_{i_1} \wedge \dots \wedge \widehat{e_{i_j}} \wedge \dots \wedge e_{i_p} \otimes m).$$

We denote the homology of this complex by $H_i(f_1, \dots, f_r; M)$.

The cohomology of the Koszul Complex admits a particularly elegant description when the chosen elements $f_1, \dots, f_r$ form an M**-regular sequence**.

Definition 53 Regular Sequence.

Let M be a module over a ring A. A sequence $f_1, \dots, f_r \in A$ is called an M-regular sequence if, for each $i = 1, \dots, r$, f_i is a non-zero divisor on $M/(f_1, \dots, f_{i-1})M$, and $M/(f_1, \dots, f_r)M \neq 0$.

Under these conditions, the Koszul complex serves as a free resolution of M/IM (where $I = (f_1, \dots, f_r)$), and its higher cohomology groups vanish.

Proposition 54 Homology of Regular Sequences.

Let A be a ring, M an A-module, and $f_1, \dots, f_r$ a sequence of elements in A. If $f_1, \dots, f_r$ forms an M-regular sequence, then:

1. $H_i(f_1, \dots, f_r; M) = 0$ for all $i > 0$.
2. $H_0(f_1, \dots, f_r; M) \cong M/(f_1, \dots, f_r)M$.

Proof. We proceed by induction on the length r of the sequence.

Base case $r = 1$: The complex is $0 \to M \xrightarrow{f_1} M \to 0$. $H_1(f_1; M) = \ker(f_1 : M \to M)$. Since f_1 is M-regular (non-zero divisor on M), the kernel is zero. $H_0(f_1; M) = \operatorname{coker}(f_1) = M/f_1M$. The assertion holds.

Inductive step: Assume the statement holds for sequences of length $r - 1$. Let $K'_\bullet = K_\bullet(f_1, \dots, f_{r-1}; M)$. The Koszul complex for r elements can be constructed as the mapping cone of the multiplication map $f_r : K'_\bullet \to K'_\bullet$. Alternatively, there is a short exact sequence of complexes:

$$0 \longrightarrow K'_\bullet \longrightarrow K_\bullet(f_1, \dots, f_r; M) \longrightarrow K'_\bullet[-1] \longrightarrow 0.$$

This induces a long exact sequence in homology:

$$\cdots \longrightarrow H_i(K'_\bullet) \xrightarrow{f_r} H_i(K'_\bullet) \longrightarrow H_i(f_1, \dots, f_r; M) \longrightarrow H_{i-1}(K'_\bullet) \xrightarrow{f_r} H_{i-1}(K'_\bullet) \longrightarrow \dots$$

By the induction hypothesis, $H_i(K'_\bullet) = 0$ for $i > 0$. For $i > 1$, the terms sandwiching $H_i(f_1, \dots, f_r; M)$ are both zero, so the homology vanishes. For $i = 1$, we have the segment:

$$0 \longrightarrow H_1(f_1, \dots, f_r; M) \longrightarrow H_0(K'_\bullet) \xrightarrow{f_r} H_0(K'_\bullet).$$

We know $H_0(K'_\bullet) \cong M/(f_1, \dots, f_{r-1})M$. The map is multiplication by f_r. Since the sequence is regular, f_r is a non-zero divisor on $M/(f_1, \dots, f_{r-1})M$. Thus the map is injective, forcing $H_1(f_1, \dots, f_r; M) = 0$.

Finally, for $i = 0$, the exact sequence gives the cokernel:

$$H_0(f_1, \dots, f_r; M) \cong H_0(K'_\bullet)/f_rH_0(K'_\bullet) \cong \frac{M/(f_1, \dots, f_{r-1})M}{f_r(M/(f_1, \dots, f_{r-1})M)} \cong M/(f_1, \dots, f_r)M.$$

The induction is complete. □

At this stage, we can obtain an explicit expression for the dualizing sheaf on a projective scheme X by employing the canonical sheaf of the ambient projective space and the exterior powers of the tangent bundle associated with it.

Theorem 55 Dualizing Sheaf for Local Complete Intersections.

Let X be a closed subscheme of codimension r in $P = \mathsf{P}^N_k$. Suppose X is a local complete intersection. Let $\mathscr{I}$ be the ideal sheaf of X in P. Then:

$$\omega_X^\circ \cong \omega_P \otimes \bigwedge^r (\mathscr{I}/\mathscr{I}^2)^\vee .$$

In particular, ω_X° is an invertible sheaf on X.

Proof. By definition, the dualizing sheaf is given by $\omega_X^\circ = \mathscr{E}xt^r_P(\mathscr{O}_X, \omega_P)$. To compute this, we work locally.

Let $U \subset P$ be an open affine subset such that the ideal sheaf $\mathscr{I}|_U$ is generated by r elements $f_1, \ldots, f_r \in A = \Gamma(U, \mathscr{O}_P)$. Since X is a local complete intersection of codimension r, for any point $x \in X \cap U$, the sequence $f_1, \ldots, f_r$ forms a regular sequence in the local ring $\mathscr{O}_{P,x}$.

Consequently, the **Koszul complex** $K_\bullet(f_1, \ldots, f_r; \mathscr{O}_P)$ provides a free resolution of $\mathscr{O}_X$ over U:

$$0 \to \bigwedge^r F \xrightarrow{d_r} \cdots \to F \xrightarrow{d_1} \mathscr{O}_P \to \mathscr{O}_X \to 0,$$

where $F \cong \mathscr{O}_P^{\oplus r}$ is the free module with basis $e_1, \ldots, e_r$ corresponding to the generators f_i.

To compute $\mathscr{E}xt^r_P(\mathscr{O}_X, \omega_P)$, we apply the functor $\mathscr{H}om_P(\cdot, \omega_P)$ to the deleted resolution $K_\bullet$. The relevant part of the dual complex is at the tail (degree r):

$$\cdots \to \mathscr{H}om(\bigwedge^{r-1} F, \omega_P) \xrightarrow{d_r^*} \mathscr{H}om(\bigwedge^r F, \omega_P) \to 0.$$

The last term identifies as:

$$\mathscr{H}om(\bigwedge^r F, \omega_P) \cong \omega_P \otimes (\bigwedge^r F)^\vee \cong \omega_P,$$

where the last isomorphism depends on the choice of the basis $e_1 \wedge \cdots \wedge e_r$. The map d_r^* is defined by the transpose of the Koszul boundary map. Since $f_1, \ldots, f_r$ are in the ideal defining X, the image of d_r^* lands in $\mathscr{I}\omega_P$. Thus, the cohomology at the r-th spot is:

$$\mathscr{E}xt^r_P(\mathscr{O}_X, \omega_P)|_U \cong \omega_P/(f_1, \ldots, f_r)\omega_P \cong \omega_P|_X.$$

The Issue of Well-definedness: The isomorphism $\phi_{\{f\}} : \mathscr{E}xt^r_P(\mathscr{O}_X, \omega_P)|_U \xrightarrow{\sim} \omega_P|_X$ constructed above depends on the choice of generators $f_1, \ldots, f_r$. Suppose $g_1, \ldots, g_r$ is another set of generators for $\mathscr{I}$ on U. Then we can write $g_i = \sum_{j=1}^r c_{ij} f_j$ for some matrix $C = (c_{ij})$ with entries in $\mathscr{O}_P$. Since both generate the same ideal locally, C is invertible, i.e., $\det(C) \in \mathscr{O}_P^\times$.

The transition between the Koszul complexes involves the exterior power of this matrix. Specifically, the top exterior power $\bigwedge^r F$ transforms by the factor $\det(C)$. This implies that the isomorphism changes by:

$$\phi_{\{g\}} = \det(C) \cdot \phi_{\{f\}}.$$

To construct an intrinsic isomorphism (independent of the basis), we compensate using the conormal sheaf. The sheaf $\mathscr{I}/\mathscr{I}^2$ is locally free of rank r on X, with local basis given by the classes of $f_1, \dots, f_r$. The dual sheaf $\mathcal{N} = (\mathscr{I}/\mathscr{I}^2)^\vee$ is generated by the dual basis $f_1^\vee, \dots, f_r^\vee$. Consider the line bundle $\bigwedge^r (\mathscr{I}/\mathscr{I}^2)^\vee$. Its local basis is $f_1^\vee \wedge \cdots \wedge f_r^\vee$. Under the change of basis to g_i, the dual basis transforms by the inverse transpose, and the top exterior power transforms by $\det(C^{-1}) = \det(C)^{-1}$.

Therefore, the tensor product

$$\omega_P|_X \otimes \bigwedge^r (\mathscr{I}/\mathscr{I}^2)^\vee$$

is invariant. The factor $\det(C)$ from the Ext computation cancels perfectly with the factor $\det(C)^{-1}$ from the determinant of the normal bundle.

Thus, we obtain a globally defined natural isomorphism:

$$\omega_X^\circ \cong \omega_P \otimes \bigwedge^r (\mathscr{I}/\mathscr{I}^2)^\vee.$$

□

When X is further restricted to be a non-singular projective variety, the dualizing sheaf coincides precisely with the canonical sheaf. We can see that, in this context, the sheaf effectively corresponds to the **volume form** on a differential manifold, which governs orientation and integration.

Corollary 56 Dualizing Sheaf for Nonsingular Varieties.

If X is a projective nonsingular variety over an algebraically closed field k, then the dualizing sheaf ω_X° is isomorphic to the canonical sheaf of differential forms $\omega_X = \bigwedge^{\dim X} \Omega_{X/k}$.

Proof. We embed X into a projective space $P = \mathbb{P}_k^N$. Since X is nonsingular, it is a local complete intersection in P. We invoke Theorem 55:

$$\omega_X^\circ \cong \omega_P \otimes \bigwedge^r (\mathscr{I}/\mathscr{I}^2)^\vee,$$

where $r = N - \dim X$ is the codimension.

Recall the definition of the canonical sheaf ω_X and the exact sequence relating the differentials of X and P (the conormal sequence):

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_{P/k} \otimes \mathscr{O}_X \to \Omega_{X/k} \to 0.$$

Taking the highest exterior powers (determinants) of this short exact sequence yields the adjunction formula:

$$\bigwedge^N (\Omega_{P/k} \otimes \mathscr{O}_X) \cong \bigwedge^r (\mathscr{I}/\mathscr{I}^2) \otimes \bigwedge^n \Omega_{X/k},$$

where $n = \dim X$. Rearranging for $\omega_X = \bigwedge^n \Omega_{X/k}$, we get:

$$\omega_X \cong (\omega_P|_X) \otimes \Big(\bigwedge^r (\mathscr{I}/\mathscr{I}^2)\Big)^\vee \cong \omega_P \otimes \bigwedge^r (\mathscr{I}/\mathscr{I}^2)^\vee.$$

Comparing this with the formula for ω_X°, we conclude that $\omega_X^\circ \cong \omega_X$. ☐

At this point, our duality theory reaches its zenith. For a **non-singular projective variety over an algebraically closed field**, the **Hodge symmetry** is elegantly manifested: specifically, the q-th cohomology of the sheaf of p-forms is strictly dual to the $(n-q)$-th cohomology of the sheaf of $(n-p)$-forms. This is not only a profound result in algebraic geometry but also a striking recurrence of classical manifolds theory within an algebraic framework.

Corollary 57 Serre Duality for Differential Forms.

Let X be a nonsingular projective variety of dimension n over an algebraically closed field k. Let $\Omega^p_{X/k} = \bigwedge^p \Omega_{X/k}$ be the sheaf of differential p-forms. Then for any $p, q = 0, \dots, n$, there is a natural isomorphism:

$$H^q(X, \Omega^p_{X/k}) \cong H^{n-q}(X, \Omega^{n-p}_{X/k})^\vee.$$

Consequently, the Hodge numbers $h^{p,q} = \dim_k H^q(X, \Omega^p_{X/k})$ satisfy the symmetry $h^{p,q} = h^{n-p,n-q}$.

Proof. Since X is nonsingular, the dualizing sheaf is the canonical sheaf $\omega_X = \Omega^n_{X/k}$. The sheaf $\Omega^p_{X/k}$ is locally free. Applying the Duality for Vector Bundles (Corollary 48), we have:

$$H^q(X, \Omega^p) \cong H^{n-q}(X, (\Omega^p)^\vee \otimes \omega_X)^\vee.$$

It remains to identify the sheaf $(\Omega^p)^\vee \otimes \omega_X$. Consider the perfect pairing given by the exterior algebra structure:

$$\Omega^p \otimes \Omega^{n-p} \longrightarrow \Omega^n = \omega_X.$$

This pairing induces an isomorphism $\Omega^{n-p} \xrightarrow{\sim} \mathscr{H}om(\Omega^p, \omega_X) \cong (\Omega^p)^\vee \otimes \omega_X$. Substituting this into the duality formula yields the result. ☐

Previously, we discussed the **arithmetic genus** in detail using the Euler characteristic, while only briefly defining the **geometric genus** as the dimension of the global sections of the canonical sheaf. Now, we can finally place them on equal footing. Under this dual relationship, the two genera are highly interconnected; most notably, in the case of algebraic curves, the arithmetic and geometric genera coincide perfectly.

Note 58 Cohomological Invariants: Genus and Irregularity.

The duality theorem provides a fundamental relation between the geometric invariants of X.

1. **Non-triviality of the Canonical Cohomology:** Since $H^0(X, \mathscr{O}_X) \cong k$, duality implies $H^n(X, \omega_X) \cong k$. The existence of this non-zero cohomology group is not obvious a priori without duality.
2. **Curves ($n = 1$):** Let $p_a(X) = \dim H^1(X, \mathscr{O}_X)$ be the *arithmetic genus* and $p_g(X) = \dim H^0(X, \omega_X)$ be the *geometric genus*. Duality implies $H^1(\mathscr{O}_X)$ is dual to $H^0(\omega_X)$.

Thus, for a nonsingular projective curve:

$$p_a(X) = p_g(X).$$

3. **Surfaces ($n = 2$):** Duality implies $H^2(X, \mathscr{O}_X)$ is dual to $H^0(X, \omega_X)$. Thus $p_g(X) = \dim H^2(X, \mathscr{O}_X)$. Recall that the arithmetic genus for a surface is defined as $p_a(X) = \chi(\mathscr{O}_X) - 1 = \dim H^2(\mathscr{O}_X) - \dim H^1(\mathscr{O}_X)$. Substituting p_g, we find:

$$p_g(X) - p_a(X) = \dim H^1(X, \mathscr{O}_X).$$

This difference is non-negative and is called the **irregularity** of the surface, denoted by $q = \dim H^1(X, \mathscr{O}_X)$.

Intuition 59 The Abstract Trace vs. Residues.

A subtle weakness of the proof of Theorem 47 is that the trace map $t : H^n(X, \omega_X) \to k$ is constructed abstractly via the embedding into P^N. We know it exists and is unique up to isomorphism, but we lack an intrinsic local description of how a differential form "integrates" to a number.

In the specific case of curves ($n = 1$), there is a more concrete approach using **Residues**. One can define the local residue of a differential form at a point, and the Sum of Residues Theorem provides an explicit realization of the trace map. This classical perspective offers a powerful computational tool that the general abstract theory obscures.

As the concluding part of this section and chapter, we will employ duality theory to abstract and generalize the classical concept of the **residue** from complex analysis. We begin by defining the "residue" from a purely algebraic perspective, where it is realized as a specialized form of the **trace map**.

Proposition 60 Existence of Residues on Curves.

Let X be a complete nonsingular curve over an algebraically closed field k. Let K be the function field of X. Let Ω_K be the module of differentials of K over k.

For each closed point $P \in X$, there exists a unique k-linear map

$$\mathrm{res}_P : \Omega_K \to k$$

satisfying the following properties:

(a) $\mathrm{res}_P(\tau) = 0$ for all regular differentials $\tau \in \Omega_P$ (where Ω_P is the stalk of Ω_X at P).

(b) $\mathrm{res}_P(f^n\, df) = 0$ for all $f \in K^\times$ and all integers $n \neq -1$.

(c) $\mathrm{res}_P(f^{-1}\, df) = v_P(f) \cdot 1$, where v_P is the valuation associated to the discrete valuation ring $\mathscr{O}_P$.

Proof. The uniqueness of such a map, if it exists, can be demonstrated by explicit calculation using a local parameter. Let $t \in \mathscr{O}_P$ be a uniformizing parameter. Then dt is a generator for Ω_K as a K-vector space. Thus, any $\tau \in \Omega_K$ can be written as $\tau = g\, dt$ for some $g \in K$.

Since $\mathscr{O}_P$ is a valuation ring, we can expand g as a Laurent series:

$$g = \sum_{i \ll 0} a_i t^i + h,$$

where $a_i \in k$, the sum is finite, and $h \in \mathscr{O}_P$. Substituting this into the expression for τ:

$$\tau = \left(\sum a_i t^i \right) dt + h\, dt.$$

Using the linearity of res_P and property (a), $\mathrm{res}_P(h\, dt) = 0$. For the terms $t^i dt$, property (b) implies $\mathrm{res}_P(t^i dt) = 0$ for $i \neq -1$. For the term $t^{-1} dt$, property (c) implies $\mathrm{res}_P(t^{-1} dt) = v_P(t) = 1$. Therefore, we must have:

$$\mathrm{res}_P(\tau) = a_{-1}.$$

This formula shows that res_P is uniquely determined by the coefficient a_{-1} of the local expansion.

The proof of existence is more involved. One approach (Serre, Algebraic Groups and Class Fields) is to take the formula $\mathrm{res}_P(\tau) = a_{-1}$ as the definition. However, proving that this definition is independent of the choice of the uniformizing parameter t is non-trivial, especially in characteristic $p > 0$. Another approach (Tate) constructs the residue map intrinsically using traces of certain infinite-dimensional linear operators on K. □

Following this, we will state the **Residue Theorem**. In the context of algebraic geometry, this theorem manifests as an elegant global equilibrium: for any global section of the sheaf of differentials, the sum of its local residues across all points on the variety must vanish.

Theorem 61 The Residue Theorem.

Let X be a complete nonsingular curve. For any differential form $\tau \in \Omega_K$, the sum of residues over all closed points vanishes:

$$\sum_{P \in X} \mathrm{res}_P(\tau) = 0.$$

Proof. Historically, this theorem is first proved for the projective line P^1 by explicit calculation using the standard coordinate and the coordinate at infinity. The general case for a curve X is then obtained by considering a finite morphism $\pi : X \to \mathsf{P}^1$ and analyzing the behavior of residues under the trace of differential forms.

Connection to Serre Duality (The Explicit Trace Map): The Residue Theorem allows us to construct the abstract trace map $t : H^1(X, \Omega_X) \longrightarrow k$ explicitly. Consider the constant sheaf $\mathscr{K}_X$ associated to the function field K. We have the standard exact sequence involving the sheaf of rational functions modulo regular functions:

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{K}_X \longrightarrow \mathscr{K}_X/\mathscr{O}_X \longrightarrow 0.$$

Here $\mathscr{K}_X$ is a flasque sheaf. The quotient is supported on points and decomposes as:

$$\mathscr{K}_X/\mathscr{O}_X \cong \bigoplus_{P \in X} i_*(K/\mathscr{O}_P).$$

Tensoring this sequence with the locally free sheaf Ω_X (which is flat), we obtain a flasque resolution of Ω_X:

$$0 \longrightarrow \Omega_X \longrightarrow \Omega_X \otimes \mathscr{K}_X \longrightarrow \bigoplus_{P \in X} i_*(\Omega_K/\Omega_P) \longrightarrow 0.$$

Taking the long exact sequence of cohomology, and noting that H^1 of flasque sheaves vanishes, we get:

$$\Omega_K \longrightarrow \bigoplus_{P \in X} (\Omega_K/\Omega_P) \longrightarrow H^1(X, \Omega_X) \longrightarrow 0.$$

The term in the middle represents collections of principal parts of differential forms. We define a linear map to k by summing the residues:

$$\text{Sum} : \bigoplus_{P \in X} (\Omega_K/\Omega_P) \longrightarrow k, \quad (\tau_P)_P \longmapsto \sum_{P \in X} \text{res}_P(\tau_P).$$

By the **Residue Theorem**, the image of a global rational differential form $\tau \in \Omega_K$ under the first map is zero (since the sum of its residues is zero). Therefore, the map "Sum" factors through the quotient $H^1(X, \Omega_X)$.

This induced map $t : H^1(X, \Omega_X) \to k$ is precisely the trace map required by the duality theorem, now presented in an explicit, computable form. □

Note 62 The Kodaira Vanishing Theorem.

Our discussion of cohomology on projective varieties is completed by mentioning a fundamental result concerning complex geometry.

Let X be a projective nonsingular variety of dimension n over the complex numbers $\mathbb{C}$. Let $\mathscr{L}$ be an ample invertible sheaf on X. Then:

(a) $H^i(X, \mathscr{L} \otimes \omega_X) = 0$ for all $i > 0$.

(b) $H^i(X, \mathscr{L}^{-1}) = 0$ for all $i < n$.

These two statements are equivalent to each other by Serre Duality. Specifically, $H^i(X, \mathscr{L}^{-1})$ is dual to $H^{n-i}(X, \mathscr{L}^{-1\vee} \otimes \omega_X) = H^{n-i}(X, \mathscr{L} \otimes \omega_X)$. If (a) holds, then for $i < n$, the index $n - i > 0$, so the group vanishes.

Remark: The original proof uses methods of complex analytic differential geometry (Kähler metrics and harmonic forms). While algebraic proofs exist for certain cases, Raynaud has shown that this theorem generally **fails** over fields of characteristic $p > 0$.

CHAPTER 10

Curves and Surfaces

10.1 What is Curves?

10.2 Riemann-Roch Theorem